Noncommutative Schwarz lemma and Pick–Julia theorems for generalized derivations in Q, Q* and Schatten-von Neumann ideals of compact operators

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Abstract. If a derivation $AX - XB$ is a trace class ($\mathcal{C}_1(\mathcal{H})$) operator for some bounded operator $X \in \mathcal{B}(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$, then for all holomorphic function f, which maps the open unit disc $\mathbb{D} \subset \mathbb{C}$ into itself, we have shown that $f(A)X - Xf(B) \in \mathcal{C}_1(\mathcal{H})$ and

$$\begin{aligned}&\big\|\sqrt{I - A^*A}\big(f(A)X - Xf(B)\big)\sqrt{I - BB^*}\big\|_1 \\ &\leqslant \big\|\sqrt{I - f(A)^*f(A)}\,(AX - XB)\sqrt{I - f(B)f(B)^*}\big\|_1.\end{aligned}$$

If C and D are strictly accretive operators on $\mathcal{H}$ and at least one of them is normal, such that $CX - XD \in \mathcal{C}_\Psi(\mathcal{H})$ for some $X \in \mathcal{B}(\mathcal{H})$ and Q^* symmetrically norming function Ψ, then for all holomorphic functions h, mapping the open right half (complex) plane into itself, we have $h(C)X - Xh(D) \in \mathcal{C}_\Psi(\mathcal{H})$, satisfying

$$\begin{aligned}&\big\|(C^* + C)^{1/2}\big(h(C)X - Xh(D)\big)(D + D^*)^{1/2}\big\|_\Psi \\ &\leqslant \big\|\big(h(C)^* + h(C)\big)^{1/2}(CX - XD)\big(h(D) + h(D)^*\big)^{1/2}\big\|_\Psi.\end{aligned}$$

If $1 \leqslant q, r, s \leqslant +\infty$ and $p \geqslant 2, A, B, X \in \mathcal{B}(\mathcal{H})$ and A, B are strict contractions satisfying the condition $AX - XB \in \mathcal{C}_s(\mathcal{H})$, then for all holomorphic functions g, mapping the open unit disc into the open right half (complex) plane, $g(A)X - Xg(B) \in \mathcal{C}_s(\mathcal{H})$, satisfying Schatten-von Neumann s-norms ($\|\cdot\|_s$) inequality

$$\begin{aligned}&\Big\|\Big|\big(g(A)^* + g(A)\big)^{\frac{1}{2}}(I - A^*A)^{\frac{1}{2}}\Big|^{\frac{1}{q}-1}(I - A^*A)^{\frac{1}{2}}\big(g(A)X - Xg(B)\big) \\ &\quad\times (I - BB^*)^{\frac{1}{2}}\Big|\big(g(B) + g(B)^*\big)^{\frac{1}{2}}(I - BB^*)^{\frac{1}{2}}\Big|^{\frac{1}{r}-1}\Big\|_\Psi \\ &\leqslant \Big\|\Big|\big(g(A)^* + g(A)\big)^{\frac{1}{2}}(I - AA^*)^{\frac{1}{2}}\Big|^{\frac{1}{q}}(I - AA^*)^{-\frac{1}{2}}(AX - XB) \\ &\quad\times (I - B^*B)^{-\frac{1}{2}}\Big|\big(g(B) + g(B)^*\big)^{\frac{1}{2}}(I - B^*B)^{\frac{1}{2}}\Big|^{\frac{1}{r}}\Big\|_s.\end{aligned}$$

Various other variants of some new Pick–Julia type norm and operator inequalities are also obtained, they complement the well-known Pick–Julia theorems for operators, obtained by Ky Fan, Ando and others, and they also extend these theorems to the field of norm ideals of compact operators, including Q, Q* and Schatten–von Neumann ideals.

Keywords: Norm inequalities; Schatten–von Neumann ideals; Q and Q* norms

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