Fifty Years of the Ćirić Fixed Point Theorem on Quasi-Contractions

Ivan Aranđelović

University of Belgrade, Faculty of Mechanical Engineering,
11000 Beograd, Kraljice Marije 16, Serbia
e-mail: iarandjelovic@mas.bg.ac.rs

Zoran Mitrović

University of Banja Luka, Faculty of Electrical Engineering,
78000 Banja Luka, Patre 5, Bosnia and Herzegovina
e-mail: zoran.mitrovic@etf.unibl.org

Abstract. The notion of quasi-contractions was introduced by Lj. B. Ćirić [2]. The paper [2] has been cited over 700 times up to now. The result on existence and uniqueness of fixed point for quasi-contractions by Ćirić is one of the most general results in the metrical fixed point theory (see [1]).

In this talk (paper), we present a short survey of development of theory of quasi-contractive mappings, their generalizations and applications. Further, using quasi-contractive inequality with the non-linear comparison function, we obtain a result on existence and uniqueness of the common fixed point for hybrid pair of single-valued and multi-valued mapping defined on b-metric spaces. Our result generalizes earlier results obtained in [3] and [4]. Also, we give the application of the obtained results to dynamic systems.

Let (X, d, s) be a b-metric space and $\mathcal{B}(X)$ is a family of all nonempty bounded subsets of X. We shall use the function $\delta : \mathcal{B}(X) \times \mathcal{B}(X) \to [0, +\infty)$ defined by $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$, for any $A, B \in \mathcal{B}(X)$.

Theorem 1. *Let X be an arbitrary nonempty set, (Y, d, s) be a b-metric space, $F : X \to \mathcal{B}(Y)$ and $g : X \to Y$ be multi-valued and single-valued functions respectively. Suppose that $F(X) \subseteq g(X)$, $g(X)$ is a complete subspace of Y and that there exists the function $\varphi : [0, +\infty) \to [0, +\infty)$ such that:*

Suppose that there exists functions $\varphi_i : [0, +\infty) \to [0, +\infty)$, $i \in \{1, \dots, 5\}$ such that:

(a) $\varphi_i(0) = 0$, $\varphi_i(r) < r$ for all $r > 0$,

(b) $\lim\limits_{x \to +\infty}(x - \varphi_i(x)) = +\infty$, $\overline{\lim}_{t \to r-}\varphi(t) < r$, $\overline{\lim}_{t \to r+}\varphi_i(t) \leq r$ for any $r > 0$,

(c) if $\overline{\lim\limits_{t \to s+}}\varphi(t) = s$ for some $s > 0$, then there exists $\varepsilon > 0$ such that $\varphi(t) = s$ for any $t \in (s, s + \varepsilon)$, and

(d) $\delta(Fx, Fy) \leq \max\{\varphi_1(\delta(gx, gy)), \varphi_2(\delta(gx, Fx)), \varphi_3(\delta(gy, Fy)), \varphi_4(\delta(gx, Fy)), \varphi_5(\delta(Fx, gy)\}$, for all $x, y \in X$.

Then F and g have the unique coincidence point $z \in Y$ such that z is the limit of every Jungck sequence defined by F and g. Moreover, if $X = Y$ and F and g are weakly compatible, then z is the unique common strict fixed point of F and g.

Keywords: fixed point; quasi-contractions; b-metric space.

References

[1] **I. Aranđelović, M. Mateljević.** Ljubomir B. Ćirić (1935-2016). *Filomat*, 2017, 31:11, 3035–3040.

[2] **Lj. B. Ćirić.** A generalization of Banach's contraction principle. *Proc. Amer. Math. Soc.*, 1974, 45, 267 – 273.

[3] **Lj. B. Ćirić, J. S. Ume.** Multi-valued mappings on metric spaces, Mathematica (Cluj) 46 (55) (2004) No. 1, 41–46.

[4] **N. Savanović, I. Aranđelović, Z. Mitrović.** The results on coincidence and common fixed points for a new type multi valued mappings in b-metric spaces. *Mathematics*, 2022, 10, 856.