Estimates for the diameter of planar Brownian motion

Milica Karapetrović

School of Electrical Engineering, University of Belgrade, Bulevar kralja Aleksandra 73
e-mail: milica@etf.rs

Abstract. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If X is a random variable, then the expectation of X will be denoted by $\mathbb{E}X$ with respect to the given probability $\mathbb{P}$. Let $\mathrm{B}(t)$, where $t \in [0,1]$, be a standard planar Brownian motion. For $0 \leqslant \theta \leqslant \pi$ we introduce the parametrized range function r given by

$$\mathrm{r}(\theta) = \sup_{t\in[0,1]} (\mathrm{B}(t) \cdot \mathrm{e}_\theta) - \inf_{t\in[0,1]} (\mathrm{B}(t) \cdot \mathrm{e}_\theta),$$

with e_θ being the unit vector $(\cos\theta, \sin\theta)$. We find the common distribution function F of the random variables $\mathrm{r}(\theta)$. Namely, we prove that

$$F(x) = 8 \sum_{n=1}^{\infty} \left(\frac{1}{x^2} + \frac{1}{(2n-1)^2\pi^2} \right) \exp\left(-\frac{(2n-1)^2\pi^2}{2x^2} \right),$$

for every $x > 0$.

Let d be the diameter of the set $\mathrm{B}[0,1]$, that is $\mathrm{d} = \operatorname{diam} \mathrm{B}[0,1] = \sup\{\|\mathrm{B}(t) - \mathrm{B}(s)\| : t, s \in [0,1]\}$, where $\|\cdot\|$ denotes the two-dimensional Euclidean norm. It is known that

$$1.601 \leqslant \mathbb{E}\mathrm{d} \leqslant 2.355.$$

We provide better lower bound for the expected diameter of the set $\mathrm{B}[0,1]$. Namely, we have the following result

$$\mathbb{E}\mathrm{d} \geqslant 1.856.$$

Keywords: Brownian motion; Diameter; Distribution.

References

[1] **M. Jovalekić.** Lower bound for the diameter of planar Brownian motion. *Bull. Math. Soc. Math. Roumanie*, 2021, 64, 283 - 286.

[2] **J. McRedmond, C. Xu.** On the expected diameter of planar Brownian motion. *Stat. Prob. Lett.*, 2017, 130, 1 - 4.