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# КЊИГА АПСТРАКТА



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МАТЕМАТИЧКИ ФАКУЛТЕТ

**КЊИГА АПСТРАКТА**  
XV СРПСКИ МАТЕМАТИЧКИ КОНГРЕС  
19–22. јуна 2024.  
Београд, Србија

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Универзитет у Београду  
Математички факултет

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# Организација

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## Поштовани учесници и гости Конгреса, драги колеге и пријатељи математике,

Са великим задовољством и поносом представљамо 15. Српски математички конгрес, национални скуп са међународним учешћем, који се одржава у нашем прелепом граду Београду, од 19. до 22. јуна 2024. године, у организацији Математичког факултета, Универзитета у Београду и Друштва математичара Србије, у сарадњи са бројним научно-истраживачким организацијама природно-математичке групације из целе Србије.

Овај Конгрес представља не само главни научни скуп у области математике у нашој земљи ове године, већ и један изузетан догађај на међународној научној сцени у области математичких и рачунарских наука. Овде смо да бисмо обогатили наше знање, разменили идеје, створили плодне сарадње и партнерства и оснажили научни подмладак у овим научним областима.

Желимо да изразимо своју дубоку захвалност Министарству просвете, као и Министарству науке, технолошког развоја и иновација Републике Србије, компанијама Телеком Србија, Банка Intesa и DataArt чија је подршка била од изузетног значаја за успешно организовање овог Конгреса. Без њихове подршке, било веома тешко реализовати овако велики научни скуп.

На овом Конгресу, ми смо сви једнаки - из различитих делова света, са различитим искуствима и амбицијама, али са једним заједничким циљем - да допринесемо научном напретку и развоју математике и рачунарских наука у Србији и региону.

Нека ови дани буду испуњени новим открићима, конструктивним разменама идеја и резултата, као и инспиративним научним дискусијама. Желимо Вам пријатан и успешан боравак у нашем Београду на 15. Српском математичком конгресу.

Организациони одбор 15. Српског математичког конгреса

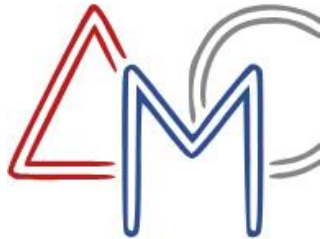
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## КОНГРЕС СУ ПОДРЖАЛИ



Министарство просвете  
Министарство науке, технолошког развоја и иновација



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## Пленарна предавања

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## Geometry of Harmonic Maps

Vladimir Marković

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**Abstract.** Harmonic maps between manifolds play a prominent role in geometry. In the finite volume they can be constructed as limits of heat flows. In this talk I will consider the non-compact case where the heat flow methods do not work so well. I will state and discuss basic existence and uniqueness results including the recent solution of the Schoen conjecture.

## History of the Constraint Satisfaction Problem: Motivation, Techniques, Solution and Beyond

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**Abstract.** I will start by reviewing the basic premises of the Descriptive Complexity (1980s) which led to the formulations of the Constraint Satisfaction Problem and the Dichotomy Conjecture on its complexity. Further, I will describe the methods which were developed in 2000s and 2010s, in various areas of mathematics, to resolve that conjecture. Next I will give the basic ideas used in the two proofs of the Dichotomy Conjecture (2017). Finally, the conclusion of the lecture is a review of the three directions of further research, of which I am personally most interested in the return to Descriptive Complexity, into more general settings than the Constraint Satisfaction, with the aim at eventually describing the class P.



## Supergeometry and supersymmetries: an introduction

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**Abstract.** A “geometrical space” is a topological space endowed with a sheaf of commutative algebras: choosing different local models for the space yields different kinds of geometry: (real) differential geometry, (complex) analytic/holomorphic geometry, and algebraic geometry of schemes. In the last case, the sheaf-theoretic perspective can also be replaced by functorial point of view, where each space is replaced by its functor of points. In these geometries, symmetries are encoded into spaces which are also groups: this yields (real or complex) Lie groups and group-schemes; when restricting to infinitesimal symmetries, one considers Lie algebras instead.

Supergeometry is the outcome of playing the above game with commutative algebras replaced by commutative *superalgebras* — i.e.,  $\mathbb{Z}_2$ -graded algebras whose homogeneous elements commute or anticommute with each other. Their symmetries are formalized by the notions of (real or complex) *Lie supergroup* and of *supergroup-scheme*, and infinitesimal symmetries by that of *Lie superalgebra*.

In this talk I will introduce the basics of supergeometry and its symmetries, mainly stressing their algebraic version, i.e. the *algebraic* supergeometry of superschemes and supergroup-schemes, following the functor of points perspective. Besides introducing the basic definitions, I will focus onto the link between supergroups and Lie superalgebras, and finally I will present the technique of studying supergroups via super Harish-Chandra pairs.

**Keywords:** Superalgebra, Supergeometry, Algebraic Supergroups, Lie Superalgebras.

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## Existence of Large-Data Global Weak Solutions to Navier–Stokes–Fokker–Planck Systems

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**Abstract.** Since the pioneering contributions of Werner Kuhn, Hans Kramers and other scientists working at the interface of polymer chemistry and statistical physics during the first half of the twentieth century, kinetic models have been widely and successfully used to describe the motion of polymeric fluids. The aim of this talk is to review recent results concerning the mathematical analysis of these models. We focus in particular on questions of existence of large-data global-in-time weak solutions to kinetic models of dilute polymeric fluids – a system of nonlinear partial differential equations involving the compressible or incompressible Navier–Stokes equations, modelling the evolution of the velocity field and the pressure, coupled to the Fokker–Planck equation satisfied by the probability density function for the random configuration vectors associated with the directions of the backbones of noninteracting polymer molecules suspended in a Newtonian fluid. We shall highlight some nontrivial open problems related to the breakdown of weak compactness in  $L^1$  and the appearance of the divergence of a symmetric positive semi-definite matrix-valued defect measure in the balance of linear momentum equation in the Navier–Stokes system. The key results discussed in the talk may be found in [1]–[5].

**Keywords:** Navier–Stokes–Fokker–Planck systems; global weak solutions; polymeric fluids.

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## Geometry of Harmonic Maps

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**Abstract.** Volume is a useful invariant of a hyperbolic 3-manifold which can be estimated from its polyhedral decomposition. By Belletti theorem [1] the exact upper bound for the volumes of generalized hyperbolic polyhedra with the same one-dimensional skeleton  $G$  equals the volume of an ideal right-angled hyperbolic polyhedron whose one-dimensional skeleton is the medial graph for  $G$ . We will discuss the volume bounds obtained in [2] for ideal right-angled hyperbolic polyhedra and in [3] for generalized hyperbolic polyhedra. The bounds depend linearly of the number of edges of a polyhedron. As an application we get the new upper bound for volumes of hyperbolic complements of links with more than eight twists in diagrams.

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## Hölder and Lipschitz continuity of mapping in the Sobolev classes $W^{1,p}(G)$

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**Abstract.** In this research among the other things, we combine the method of the flattening the boundary with the Sobolev-Riesz embeddings theorem. Our results include domains which are locally good Greenian and  $C^1$  domains.

In particular we study the Dirichlet problem for generalized harmonic mappings and boundary behaviour of partial derivatives for solutions to certain Laplacian-gradient inequalities and generalized harmonic mappings.

**Keywords:** The Sobolev-Riesz embeddings theorem; Laplacian-gradient inequalities

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## On Random Environment Integer-valued Autoregressive Models

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**Abstract.** The main problem in modeling counting data is naturally defining the most suitable time series model, which would be the best possible representation of the observed real-life data. The data which we are interested are obtained by registering the correlated integer values of the particular phenomenon, or simply as a result of counting realizations of some phenomena, or even the elements of an observed population, in successive time intervals. The aforementioned data can be found in many fields of natural sciences and humanities including medicine, economics, finance, telecommunications, criminology, sports, etc. In order to model such data as well as possible, mathematicians used at first autoregressive time series with continuous marginal distributions. This gave acceptable results only when it comes to phenomena that generate extremely high realization values, where round-off error is negligible. On the other hand, in situations when observations registered over time are not that high (lower than  $10^6$ ), previously mentioned models cannot be used successfully. Slightly better results were achieved by involving Markov chains into the modeling procedure, as described in [ 1]. Unusually big number of parameters was the key obstacle to this idea. Few years later, several Discrete Autoregressive models of Moving Average (*DARMA*), based on well known *ARMA* models, were defined in [ 2]. These models gave even better results. Finally, in mid 1980s, [ 3] and [ 4] introduced in different ways an Integer-valued Autoregressive model of order 1 (*INAR*(1)), based on the binomial thinning operator. This newly introduced model gave a foundation to the modern and contemporary approach in modeling the counting data sets.

However, *INAR* models based on binomial thinning were not suitable for describing, for example, population sizes through time in situations where it can change not only by their members disappearance, but also through their interactions or reproductions. With the aim of this kind model improvement, a significant breakthrough was made in [ 5] with the introduction of the negative binomial thinning operator and the so-called *NGINAR* model with a geometric marginal distribution. After that, a large number of variations and generalizations of *INAR* models, in terms of their orders, thinning operators, dimensionality and marginal distributions, have emerged in order to achieve more adequate fitting of counting data of various kinds.

All mentioned models were stationary. But very often certain non-stationary characteristics have appeared in the counting processes. So, that led us to our second milestone in *INAR* modeling, which was introducing a completely new concept of model construction by incorporating in data-fitting the effects of environment in which the counting process exists and which can change through time. We achieved this in [ 6] by introducing additional (controlling) process (Markov chain) which by taking different values of its random variables, so-called random states, in different periods of time, affects counting model values, by directly defining its marginal distribution parameter values. Later, in [ 7] and [ 8] some generalizations in respect of controlling the order of the model and correlation parameter value were presented using the same random state controlling process which was obtained by some process values clustering algorithms. Besides using well-known *k*-means clustering algorithm we have also tried to introduce more advanced clustering mechanism for our specific non-stationary *INAR* values, which was presented in [ 9] and [ 10]. Some bivariate random environment *INAR* models, also driven by a single process of random states, were successfully introduced in [ 11]

However, we have now realized that although the parameter values of the marginal distribution are intuitively driven by the random process defined by clustering the counting process values, that might not be the case with correlation parameter  $\alpha$ . Namely, autocorrelation of the process should also depend on environment conditions, but not necessarily in the same way as marginals. Therefore, we introduce a new model with two control processes, where a separate and independent Markov process will determine the correlation in the model. It is also obtained by clustering, but not the values of the *INAR* process, but its

sample autocorrelation functions. On this matter we have results given in [ 12].

**Keywords:** Random environment; INAR; RrNGINAR; negative binomial thinning; counting process.

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## Аласовим трагом до наших дана

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**Апстракт.** Представљамо генезу и развој геометријских идеја и техника у изучавању покретних сингуларитета обичних алгебарских диференцијалних једначина. Тако долазимо до радова Михаила Петровића Аласа и специјално, до његових геометријских идеја изражених помоћу полигоналног метода са краја 19. века. Сличан, мада различит приступ се појавио, испоставља се, и код Хенрија Фајна. Полигонални метод се може видети као уопштење Њутн-Пизоовог метода пренетог са алгебарских на диференцијалне једначине. Ове Аласове идеје, иако фундаменталне, остале су потпуно непримећене у савременој литератури, а и пре тога. Зато указујемо на њихов значај са тачке гледишта модерне математике. Уједно дајемо и краћи преглед најновијих резултата српских математичара о геометријским аспектима аналитичке теорије диференцијалних једначина.

**Кључне речи:** Петровићеви и Фајнови полигони; покретни сингуларитети; Пенлевеове једначине.

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## Неки сусрети вештачке интелигенције, блокчејн технологије и математике

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**Апстракт.** Разматрају се неке илустративне међусобне везе између нових актуелних и на математици заснованих дисциплина и традиционалних математичких приступа анализи. Генеративна вештачка интелигенција је отворила ново поглавље у свеукупном развоју, а блокчејн технологија се појављује као један од битних ослонаца. Указује се на значај који имају велике базе података и на значај веродостојности као и потребу заштите приватности и обезбеђивање сигурности. Са друге стране, поред указивања на математичке основе и машинског учења и техника за блокчејн технологију, илустративно се указује на примене традиционалних математичких метода за остваривање жељених карактеристика.

## Proofs-as-programs: from logic to AI

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**Abstract.** The proofs-as-programs correspondence is a foundational concept that connects logic and computation. The origins of this idea can be traced back to the relationship between logic and lambda calculus. Extensions to various logical and computational systems highlights its versatility and broad applicability across different domains of mathematics and computer science. It is at the heart of formal verification of mathematical proofs.

In this talk, we give an overview of this correspondence in different frameworks of computation and communication in distributed systems. The focus is on recent results which lead to safe orchestrations of federated (machine) learning algorithms.



## О значају математике у систему обавезног образовања

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**Апстракт.** У историји људске цивилизације развој математике почиње од решавања практичних животних задатака: бројање стада, трговина, изградња склоништа, робна размена. Међутим, од времена Античке Грчке све до данас, већ преко 2500 година, математика заузима посебно место у развоју људске мисли. Поред примењених задатака, она непрекидно учествује у развоју мисаоних процеса људског појединца и целог човечанства. У савремено доба, настава математике је незаобилазни стуб ослонац општег обавезног образовања који служи пре свега за развој логичког мишљења. Математику треба учити већ и због тога што она сређује мисли, кажу да је у наставном плану кадетске војне школе написао руски природњак Михаил Ломоносов \*.

Глобалистичке тенденције контроле понашања људске популације преко масмедија, које карактеришу постмодерно западно друштво и које је у свом ремек-делу „1984” предвидео енглески писац Џ. Орвел, теже да ограниче моћ логичног људског размишљања и понашања. Као природна последица тога, све је јача тежња владајуће класе да смањи значај и ниво знања математике у школи, под разноразним изговорима: прагматичним, инклузивним, егалитарним. Ова опасност се већ више од двадесет година интензивно надвија и над српским образовањем. А резултат ће свуда бити, како је приметио руски математичар Владимир Арнолд, такав да ће бродови тонати и небодери горети \*\*. Мостови се већ руше.

У овом предавању ће бити изложено више примера знања математике у школи некад и сад, примери погрешних методичких приступа у основној и средњој школи и предлози побољшања, уз много цитата умних математичара. Ево једног за крај. Како је једном рекао Херман Вајл, бављење математиком је креативна и оригинална човекова активност, попут језика или музике, која пркоси пуној објективној рационализацији \*\*\*. И зато АИ никада неће доказати Риманову хипотезу.

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\* „Математику уже затем учить надо, что она ум в порядок приводит.” Ова се изрека приписује Ломоносову, али за то нема писане потврде.

\*\* „... что не только атомоходы будут тонуть, но и все остальное, не только башня будет гореть, но и остальное тоже” (В.И. Арнолд: Путешествие в хаосе, Наука и жизнь 12:2000, <https://www.nkj.ru/archive/articles/5174/>)

\*\*\* „"Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization." (Hermann Weyl, Levels of Infinity: Selected Writings on Mathematics and Philosophy)

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## Област 1

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Реална и функционална анализа,  
Комплексна анализа, Глобална анализа,  
Теорија оператора, Аналитичка теорија  
бројева, Парцијалне и диференцијалне  
једначине



## Semi-Fredholm theory in unital $C^*$ -algebras

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**Abstract.** Axiomatic (semi-)Fredholm theory in unital  $C^*$ -algebras has been established by Kečkić and Lazović in [ 8] and by Ivković in [ 5]. In this talk, we shall present a new approach to this theory given in [ 6] and [ 7]. Next, we shall extend Fredholm theory in von Neumann algebras established by Breuer in [ 1] and [ 2] to spectral Fredholm theory in von Neumann algebras. We shall consider upper triangular operator matrices with coefficients in a von Neumann algebra and give the relationship between the generalized essential spectra (in the sense of Breuer) of such matrices and of their diagonal entries, thus generalizing in this setting the result by Đorđević in [ 3]. Finally, we shall introduce the concept of generalized Browder operators in von Neumann algebras and provide a characterization of these operators in terms of the property of being generalized Fredholm in the sense of Breuer and having 0 as an isolated point of the spectrum. Also, we shall illustrate this result by concrete examples. This is a continuation of the research in [ 4].

**Keywords:** Fredholm operator, Browder operator, von Neumann algebra, essential spectra

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## Norm of the Hilbert matrix operator on positively indexed weighted Bergman spaces

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**Abstract.** It is known [6] that the Hilbert matrix is bounded on weighted Bergman spaces  $A_\alpha^p$  if and only if  $1 < \alpha + 2 < p$  with the conjectured norm  $\pi / \sin \frac{(\alpha+2)\pi}{p}$ . In the case of positively indexed weighted Bergman spaces, when  $\alpha > 0$ , the conjecture is confirmed [8] for  $\alpha_0 \leq p$ , where  $\alpha_0$  is a unique zero of the function  $\Phi_\alpha(x) = 2x^2 - (4(\alpha + 2) + 1)x + 2\sqrt{\alpha + 2}\sqrt{x} + \alpha + 2$  on the interval  $(\alpha + 2, 2(\alpha + 2))$ . We prove, that if  $\alpha > 0$ , then the conjecture is valid for all

$$\frac{3\alpha}{4} + 2 + \sqrt{\left(\frac{3\alpha}{4} + 2\right)^2 - \frac{\alpha + 2}{2}} \leq p.$$

This considerably improves the best previously known result for all  $\alpha > \frac{1}{2}$ .

**Keywords:** Hilbert matrix, operator norm, weighted Bergman spaces.

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## Noncommutative Schwarz lemma and Pick–Julia theorems for generalized derivations in $\mathcal{Q}$ , $\mathcal{Q}^*$ and Schatten-von Neumann ideals of compact operators

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**Abstract.** If a derivation  $AX - XB$  is a trace class ( $\mathcal{C}_1(\mathcal{H})$ ) operator for some bounded operator  $X \in \mathcal{B}(\mathcal{H})$  acting on a Hilbert space  $\mathcal{H}$ , then for all holomorphic function  $f$ , which maps the open unit disc  $\mathbb{D} \subset \mathbb{C}$  into itself, we have shown that  $f(A)X - Xf(B) \in \mathcal{C}_1(\mathcal{H})$  and

$$\begin{aligned} & \left\| \sqrt{I - A^*A} (f(A)X - Xf(B)) \sqrt{I - BB^*} \right\|_1 \\ & \leq \left\| \sqrt{I - f(A)^*f(A)} (AX - XB) \sqrt{I - f(B)f(B)^*} \right\|_1. \end{aligned}$$

If  $C$  and  $D$  are strictly accretive operators on  $\mathcal{H}$  and at least one of them is normal, such that  $CX - XD \in \mathcal{C}_\Psi(\mathcal{H})$  for some  $X \in \mathcal{B}(\mathcal{H})$  and  $\mathcal{Q}^*$  symmetrically norming function  $\Psi$ , then for all holomorphic functions  $h$ , mapping the open right half (complex) plane into itself, we have  $h(C)X - Xh(D) \in \mathcal{C}_\Psi(\mathcal{H})$ , satisfying

$$\begin{aligned} & \left\| (C^* + C)^{1/2} (h(C)X - Xh(D)) (D + D^*)^{1/2} \right\|_\Psi \\ & \leq \left\| (h(C)^* + h(C))^{1/2} (CX - XD) (h(D) + h(D)^*)^{1/2} \right\|_\Psi. \end{aligned}$$

If  $1 \leq q, r, s \leq +\infty$  and  $p \geq 2$ ,  $A, B, X \in \mathcal{B}(\mathcal{H})$  and  $A, B$  are strict contractions satisfying the condition  $AX - XB \in \mathcal{C}_s(\mathcal{H})$ , then for all holomorphic functions  $g$ , mapping the open unit disc into the open right half (complex) plane,  $g(A)X - Xg(B) \in \mathcal{C}_s(\mathcal{H})$ , satisfying Schatten-von Neumann  $s$ -norms ( $\|\cdot\|_s$ ) inequality

$$\begin{aligned} & \left\| \left| (g(A)^* + g(A))^{\frac{1}{2}} (I - A^*A)^{\frac{1}{2}} \right|^{\frac{1}{q}-1} (I - A^*A)^{\frac{1}{2}} (g(A)X - Xg(B)) \right. \\ & \quad \left. \times (I - BB^*)^{\frac{1}{2}} \left| (g(B) + g(B)^*)^{\frac{1}{2}} (I - BB^*)^{\frac{1}{2}} \right|^{\frac{1}{r}-1} \right\|_\Psi \\ & \leq \left\| \left| (g(A)^* + g(A))^{\frac{1}{2}} (I - AA^*)^{\frac{1}{2}} \right|^{\frac{1}{q}} (I - AA^*)^{-\frac{1}{2}} (AX - XB) \right. \\ & \quad \left. \times (I - B^*B)^{-\frac{1}{2}} \left| (g(B) + g(B)^*)^{\frac{1}{2}} (I - B^*B)^{\frac{1}{2}} \right|^{\frac{1}{r}} \right\|_s. \end{aligned}$$

Various other variants of some new Pick–Julia type norm and operator inequalities are also obtained, they complement the well-known Pick–Julia theorems for operators, obtained by Ky Fan, Ando and others, and they also extend these theorems to the field of norm ideals of compact operators, including  $\mathcal{Q}$ ,  $\mathcal{Q}^*$  and Schatten–von Neumann ideals.

**Keywords:** Norm inequalities; Schatten–von Neumann ideals;  $\mathcal{Q}$  and  $\mathcal{Q}^*$  norms

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## Global regularity of Weyl pseudo-differential operators with radial symbols in each phase-space variable

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**Abstract.** We analyse a class of pseudo-differential operators in the Gelfand-Shilov setting whose Weyl symbols are radial in each phase-space variable separately. Namely, the symbols are of the form

$$a_{\vartheta}(x, \xi) := a(2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2),$$

where  $a$  is a measurable function on  $\mathbb{R}_+^d := \{r \in \mathbb{R}^d \mid r_j > 0, j = 1, \dots, d\}$  and has Gelfand-Shilov  $L^p$ -growths. We prove that the action of these pseudo-differential operators on a Gelfand-Shilov ultradistribution  $f$  can be given by a series of Hermite functions with coefficients that are explicitly computed in terms of the Laguerre coefficients of  $a$  and the Hermite coefficients of  $f$ . As a consequence, we give a characterisation of the functions  $a$  in terms of the growths of their Laguerre coefficients for which the Weyl quantisation of  $a_{\vartheta}$  are globally Gelfand-Shilov regular.

**Keywords:** Pseudo-differential operators with radial symbols, Gelfand-Shilov regularity, Hermite expansions, Laguerre expansions.

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## Shift-invariant subspaces of Sobolev spaces and wave fronts

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**Abstract.** We proceed with the study of shift-invariant spaces of Sobolev type [ 1, 2] analyzing products of their elements and connections with wave fronts [ 3]. The research was inspired by Hormander's description of wave fronts [ 4] and the paper [ 5]. Using the Fourier transform, we investigate the connection between shift-invariant subspaces  $V_s$  of Sobolev spaces  $H^s(\mathbb{R}^d)$ ,  $s \in \mathbb{R}$ , and the space of periodic distributions, and we analyze the product of elements of those spaces using the wave fronts of Sobolev type. We also give conditions under which the product of two functions from two different shift-invariant spaces belongs to some shift-invariant space. Also, using the wave front we describe the elements of the space  $V_s$  and the elements of the space of periodic tempered distributions.

This research was supported by the Science Fund of the Republic of Serbia, #GRANT No 2727, *Global and local analysis of operators and distributions* - GOALS.

**Keywords:** shift-invariant spaces; Sobolev spaces; multiplication of distributions; periodic distributions; wave fronts.

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## Fifty Years of the Ćirić Fixed Point Theorem on Quasi-Contractions

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**Abstract.** The notion of quasi-contractions was introduced by Lj. B. Ćirić [2]. The paper [2] has been cited over 700 times up to now. The result on existence and uniqueness of fixed point for quasi-contractions by Ćirić is one of the most general results in the metrical fixed point theory (see [1]).

In this talk (paper), we present a short survey of development of theory of quasi-contractive mappings, their generalizations and applications. Further, using quasi-contractive inequality with the non-linear comparison function, we obtain a result on existence and uniqueness of the common fixed point for hybrid pair of single-valued and multi-valued mapping defined on  $b$ -metric spaces. Our result generalizes earlier results obtained in [3] and [4]. Also, we give the application of the obtained results to dynamic systems.

Let  $(Y, d, s)$  be a  $b$ -metric space and  $\mathcal{B}(Y)$  is a family of all nonempty bounded subsets of  $Y$ . We will use the function  $\delta : \mathcal{B}(Y) \times \mathcal{B}(Y) \rightarrow [0, +\infty)$  defined by  $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$ , for any  $A, B \in \mathcal{B}(Y)$ .

*Theorem 1.* Let  $(Y, d, s)$  be a  $b$ -metric space,  $F : X \rightarrow \mathcal{B}(Y)$  and  $g : X \rightarrow X$  be multi-valued and single-valued functions respectively. Suppose that  $F(X) \subseteq g(X)$ ,  $g(X)$  is a complete subspace of  $Y$  and there exists the function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  such that:

- (a)  $\varphi(0) = 0$ ,  $\varphi(r) < r$  for all  $r > 0$ ,
- (b)  $\lim_{x \rightarrow +\infty} (x - \varphi(x)) = +\infty$ ,  $\overline{\lim}_{t \rightarrow r} \varphi(t) < r$  for any  $r > 0$ ,
- (c)  $\delta(F^n x, F^n y) \leq s \cdot \varphi \max\{\delta(g^i x, g^l y), \delta(g^i x, F^j x), \delta(g^k y, F^l y)\}$ ,  
for all  $i, j, k, l \in \{0, \dots, n\}$  and for all  $x, y \in X$ .

Then  $F$  and  $g$  have the unique coincidence point  $z \in Y$  such that  $z$  is the limit of every Jungck sequence defined by  $F$  and  $g$ . Moreover, if  $X = Y$  and  $F$  and  $g$  are weakly compatible, then  $z$  is the unique common strict fixed point of  $F$  and  $g$ .

**Keywords:** fixed point; quasi-contractions;  $b$ -metric space.

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## Analysis of parabolic equations with singular potentials in Kondratiev spaces of generalized stochastic processes

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**Abstract.** Stochastic parabolic equations with singular potentials arise in probabilistic modelling of uncertainty in engineering and science, for example in structural mechanics, material science, fluid dynamics, climate and turbulence modelling. In this talk we consider stochastic parabolic problems of the form

$$\partial_t U - \mathcal{L}U + Q \cdot U = F, \quad U|_{t=0} = G, \quad (1)$$

where the potential  $Q$ , the driving force  $F$  and the initial data  $G$  are generalized stochastic processes of Kondratiev type. The focus is mainly on all possible singular behaviors of potential  $Q$ , either in space and time or in random component, but also in investigation possibilities to allow for irregular coefficients in the operator  $\mathcal{L}$ . The product  $\cdot$  in above equation we interpret as the Wick product, which is introduced to give sense to the multiplication of two generalized stochastic processes, see [1]. In the analysis of these problems we combine the chaos expansion method from the white noise analysis and the concept of very weak solutions. The notion of a stochastic very weak solution of (1) is introduced and existence of unique very weak solution is proved. The questions on consistency of the stochastic very weak solutions with classical solutions are discussed.

The talk is based on recent papers [1, 2] and ongoing research.

**Keywords:** parabolic equations; stochastic parabolic equations; singular potentials; chaos expansions; very weak solutions.

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## On some fixed and common fixed point results obtained for Menger PM-spaces

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**Abstract.** In this talk, we will present some fixed point results obtained for Menger PM-spaces by employing some weaker forms of continuity and a new type of  $(\varepsilon - \delta)$  condition. Additionally, we will present the result that establishes that the completeness of the space is equivalent to the fixed point property for a larger class of mappings. On the other hand, we will show that if a pair of self-mappings that satisfy some weaker forms of commutativity defined on Menger PM spaces with a nonlinear contractive condition possess a unique common fixed point, then these mappings are orbitally continuous. Also, we will investigate whether the converse of these results holds.

**Keywords:** Menger PM-spaces; fixed point; common fixed point.

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## Schrödinger-type equations with singular potentials

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**Abstract.** Motivated by theoretical considerations of the famous double-slit experiment, we investigate notions of solutions to initial value problems for the Schrödinger equation containing irregular potentials and coefficients. Following the results of [ 1], establishing well-posedness in Colombeau generalized function spaces, we construct the very weak solution (established in [ 2]) for the problem, discuss questions of uniqueness and its consistency with weak solutions for regular coefficients.

Finally, we compare different concepts of solutions both using fundamental idea of regularising irregular coefficients and data and conclude that a combination of the methods has considerable advantages.

Results are obtained in collaboration with Christian Spreitzer from University of Vienna.

**Keywords:** generalized functions; regularization methods; Schrödinger equation

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## Weak integration of operator valued functions

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**Abstract.** For weakly\* integrable operator valued function  $\mathcal{A} : \Omega \rightarrow \mathcal{B}(\mathcal{H})$  we investigate whether its integral belongs to the appropriate ideal of compact operators. Furthermore, for a mononormalizing symmetric norming function  $\Phi$  and  $\mathcal{C}_\Phi(\mathcal{H})$ -valued weakly\* measurable function  $\mathcal{A}$ , if for every pair of orthonormal systems  $(e_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  it holds that

$$\int_{\Omega} \Phi((\mathcal{A}_t e_n, f_n)_{n \in \mathbb{N}}) d\mu(t) < +\infty,$$

we show that  $\mathcal{A}$  is Pettis  $\mathcal{C}_\Phi(\mathcal{H})$ -integrable. Additionally, in the case where  $\mu(\Omega) < +\infty$ , we provide necessary and sufficient conditions for the  $\mathcal{C}_\infty(\mathcal{H})$ -valued function  $\mathcal{A}$  to be Pettis integrable.

**Keywords:** Integration in Banach spaces; Operator valued functions; Symmetrically norming functions; Ideals of compact operators.

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## On estimating convergence for Pickard sequences in quasi $b$ -metric and rectangular quasi $b$ -metric spaces

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**Abstract.** The iterative sequence  $\{x_n\}$  with arbitrary  $x_0$  converges (under the assumptions in Banach's Fixed Point Theorem [ 1]) to the unique fixed point  $x^*$  of contraction  $T$ . Error estimates are the prior estimate

$$d(x_n, x^*) \leq \frac{\lambda^n}{1 - \lambda} d(x_0, x_1),$$

and the posterior estimate

$$d(x_n, x^*) \leq \frac{\lambda}{1 - \lambda} d(x_{n-1}, x_n).$$

Since the introduction of quasi  $b$ -metric spaces by M. H. Shah, N. Hussain [ 4], many standard results have been generalized and elaborated in some papers. Recently, Z. Mitrović, I. Arandjelović, V. Mišić, H. Aydi and B. Samet [ 3] (see also [ 2]), proved that the sequence  $\{x_n\}$  in quasi  $b$ -metric space with the contractive condition

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}),$$

for all  $n \in \mathbb{N}$ , where  $\lambda \in [0, 1)$  is Cauchy sequence.

In this paper, using the same contractive condition as in [ 3], we give an estimate of the  $d(x_n, x^*)$  for a sequence  $\{x_n\}$  in a quasi  $b$ -metric space. In addition, we give another proof for the convergence of a sequence  $\{x_n\}$ . Examples of estimation for Banach's, Kannan's, and Reich's fixed point theorems are given. Following the same idea, in the second part of talk (paper) we shall introduce notion of the rectangular quasi  $b$ -metric spaces and present estimation in this class of spaces. At the end, we give some open problems where research can be continued.

**Keywords:** Cauchy sequence; quasi  $b$ -metric spaces; rectangular quasi  $b$ -metric spaces.

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## Some applications of factorable matrices

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**Abstract.** Matrix transformations between sequence spaces, whether classical or not, are the subject of numerous publications. However, it seems that there is still a particular gap. Actually, the characterization of the matrix class  $(\ell_p, \ell_q)$  for arbitrary  $p$  and  $q$  ( $1 < p, q < \infty$ ) is still unknown. Among the results related to some special cases, the results of Grahame Bennett [1, 2, 3] and Karl-Goswin Grosse-Erdmann [4] have inspired us to think about the mentioned class of matrix transformations. Furthermore, the application of existing results can go in two directions. Firstly, we can extend the research to new sequence spaces related to  $\ell_p$  ( $1 < p < \infty$ ) and characterize certain matrix class. Further, we can find some new results that treat some known matrices as factorable.

**Keywords:** sequence spaces; matrix transformations; factorable matrices; matrix domains of triangle.

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## A note on the Toeplitz operators on weighted mixed norm spaces

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**Abstract.** Toeplitz operators  $T_\mu$ , where  $\mu$  is a finite measure, on weighted mixed norm spaces  $B_\alpha^{p,q}(\Omega)$  on a smoothly bounded domain  $\Omega \subset \mathbb{R}^n$  are studied. It is proved that  $T_\mu$  is bounded if and only if the Berezin transform of  $\mu$  is bounded on  $\Omega$  if and only if  $\mu$  is a Carleson measure on  $\Omega$  for the space  $B_\alpha^{p,q}(\Omega)$ . Analogously,  $T_\mu$  is compact if and only if the Berezin transform of  $\mu$  vanishes on  $\partial\Omega$  if and only if  $\mu$  is a vanishing Carleson measure on  $\Omega$  for the space  $B_\alpha^{p,q}(\Omega)$ . The proofs rely on the following: boundedness of Bergman projection, description of the dual space  $B_\alpha^{p,q}(\Omega)^* = B_{1-\alpha}^{p',q}(\Omega)$  and density of functions smooth up to the boundary in  $B_\alpha^{p,q}(\Omega)$

The results on Toeplitz operators and these related results are joint work with Ivana Savković (Faculty of Mechanical Engineering, University of Banja Luka).

## Partial orders in Rickart rings defined by annihilators

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**Abstract.** A matrix  $A^-$  is a generalized inverse ( $g$ -inverse) of a matrix  $A$  if  $AA^-A = A$  holds. If  $A$  and  $B$  are two matrices of the same order then we say that  $A$  is below  $B$  under the minus partial order,  $A <^- B$ , if  $AA^- = BA^-$  and  $A^-A = A^-B$  for some  $g$ -inverse  $A^-$  of matrix  $A$ , [ 3]. If we use some specific generalized inverses in this definition then we can define specific partial orders such as star, sharp, core and other partial orders. Let  $H$  be an infinite-dimensional Hilbert space and  $B(H)$  the algebra of all bounded linear operators on  $H$ . Although the minus and other orders can be extended to  $B(H)$  via  $g$ -inverses, this is not the best approach because only the operators with closed range have a generalized inverse. The minus partial order can be successful extended to  $B(H)$  using the projections, [ 6], [ 5]. In this talk we will discuss a possible way of algebraization of this approach using the annihilators. Our extensions of minus, star, sharp and core partial orders become partial order relations in the case of Rickart and Rickart star rings, [ 1], [ 2], [ 4].

**Keywords:** generalized inverse; partial orders; Rickart ring.

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## Some fixed point results for F-contractions

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**Abstract.** Based on the Banach contraction principle, many mathematicians began to formulate various contractive conditions under which a fixed point exists. D. Wardowski [1] introduced F-contractions in metric spaces and, therefore, proved a generalization of Banach contraction principle. In following ten years, Wardowski's approach, or some of its modifications, have been widely used by many researchers for obtaining new fixed-point results for single- and multivalued mappings in various kinds of spaces, such as b-metric spaces, fuzzy metric spaces, etc. In this talk, we present an overview of these investigations, modified with some improvements, especially regarding the conditions imposed on the function F entering the contractive condition.

**Keywords:** F-contraction; fixed point; Banach contraction principle.

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## Reverse and Forward Order Law for the $(b,c)$ -inverse

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**Abstract.** In a ring  $R$  it is well known that if  $a, w \in R^{-1}$ , then  $aw$  is also invertible and  $(aw)^{-1} = w^{-1}a^{-1}$ . This property is called the reverse order law. As the reverse order law holds for the classical inverse, the topic of necessary and sufficient conditions for generalized inverses became very often in the past decades.

The similar property,  $(aw)^{-1} = a^{-1}w^{-1}$ , is known as the forward order law. Contrary to the reverse order law, even if  $a$  and  $w$  are both invertible, the forward order law is not valid in general.

Here we present results for the reverse order law for the  $(b, c)$ -inverse in a unital ring. An equivalent condition for this law to hold for the  $(b, c)$ -inverse is derived. Furthermore, the forward order law for the  $(b, c)$ -inverse in a ring with a unity is introduced for different choices of  $b$  and  $c$ . Moreover, as corollaries of obtained results, equivalent conditions for the reverse order law and the forward order law for the inverse along an element are derived.

**Keywords:**  $(b, c)$ -inverse, Inverse along an element, Reverse order law, Forward order law.

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## Derivative estimates for solutions of the $(\alpha, \beta)$ -Poisson equation

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**Abstract.** We obtain Schwarz-Pick type inequalities for solutions of the  $(\alpha, \beta)$ -Poisson equation  $L_{\alpha, \beta} u = g$ , where the function  $g$  is given on the unit disc  $\mathbb{D}$  and  $\alpha$  and  $\beta$  are complex parameters satisfying  $\Re \alpha + \Re \beta > -1$ . In particular, for  $g = 0$ , we prove sharp estimate for the  $Du(0)$  for  $(\alpha, \beta)$ -harmonic functions  $u$  in terms of  $L^p$  norm of the boundary function and asymptotically sharp estimate of  $\|Du(z)\|$ . The obtained results provide a significant improvement over previous research on the subject. This is work in progress.

**Keywords:**  $(\alpha, \beta)$ -harmonic function;  $(\alpha, \beta)$ -Poisson equation; Schwarz lemma; Weighted Green function.

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## Stability of two-dimensional and three-dimensional autonomous competitive Lotka-Volterra dynamical system

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**Abstract.** A detailed analysis of the stability of equilibriums of the two-dimensional autonomous competitive Lotka-Volterra dynamical system is conducted. Necessary and sufficient conditions are determined for equilibriums to be asymptotically stable or saddles. Necessary and sufficient conditions are determined for the observed dynamical system to have no equilibriums. Moreover, bifurcation analysis of that dynamical system is obtained. Concretely, we obtained that two transcritical bifurcations occur in the observed dynamical system. One transcritical bifurcation occurs for  $d_{12} > 0$ ,  $d_{112} > 0$ ,  $d_{122} = 0$ , while the other transcritical bifurcation arises for  $d_{12} > 0$ ,  $d_{112} = 0$ ,  $d_{122} < 0$ , where  $d_{12}$  is the main determinant of the matrix of the observed system, while  $d_{112}$  and  $d_{122}$  are its minors. Furthermore, stability of equilibriums of the three-dimensional autonomous competitive Lotka-Volterra dynamical system is inspected.

**Keywords:** Lotka-Volterra; dynamical systems; stability; equilibriums; bifurcations.

## On some recent results regarding Laplace transform

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**Abstract.** As a common-knowledge procedure, Laplace transform is used in applied mathematics, physics, engineering and computation in general. By giving a comparative overview of Laplace transforms in real space setting, Banach spaces, Banach algebras and Banach modules, we will emphasize the necessity of its studying. This lecture will firstly analyze Laplace transform procedure proposed earlier by Gautchi and Milovanovic. The technique mentioned will next be used to develop appropriate method both in Banach spaces and unital commutative Banach algebras, as well as in Banach modules. Secondly, we will discuss the existence of originals (the inverse Laplace transforms) of harmonic and hyper-harmonic sequences (in accordance with regularity criteria for Sylvester modular equation) and use these results to determine the sums of some slowly convergent Banach-algebra-valued series. These findings are obtained through joint work with Bogdan Djordjevic, and in solo work done by lecturer. [ 1]

**Keywords:** Laplace transform, Banach algebras, Banach modules, Sylvester equation, Abstract series

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## On relations induced by $(b, c)$ -inverses in rings

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**Abstract.** Motivated by results on minus, sharp, core and dual core partial orders, we define a binary relation in a ring, induced by  $(b, c)$ -inverses, in a quite similar way as mentioned partial orders. We prove that this relation is an equivalence relation and we investigate some of its properties.

**Keywords:**  $(b, c)$ -inverse; equivalence relation; ring.

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## Lipschitz continuity for functions satisfying $\alpha$ Laplacian-gradient inequality

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**Abstract.** We say that function  $f$  satisfies the  $\alpha$  Laplacian-gradient inequality on the unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  if there exist positive constants  $a, b$  such that

$$(1 - |z|^2)^\alpha |\overline{L_\alpha} f(z)| \leq a |\nabla f(z)|^2 + b \text{ for every } z \in \mathbb{D}.$$

Here  $\overline{L_\alpha} = D_{\bar{z}}((1 - |z|^2)^{-\alpha} D_z)$ . Let  $G$  be domains in  $\mathbb{C}$  with  $C^2$  boundary. Based by One of results obtained in [ 1] we will prove that, if a quasiconformal diffeomorphism  $f : \mathbb{D} \rightarrow G$  satisfies the  $\alpha$  Laplace-gradient inequality, then  $f$  is Lipschitz. The proof of this result is based on the Flattening the boundary method, with some use of continuity properties of Riesz potentials. Method used for proving the main result of this article is sometimes referred as Bootstrap argument.

**Keywords:** Poisson kernel; Green function; Generalized harmonic functions.

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## $L^p$ norm of truncated Riesz transform and an improved dimension-free $L^p$ estimate for maximal Riesz transform

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**Abstract.** We prove that the  $L^p(\mathbb{R}^d)$  norm of the maximal truncated Riesz transform in terms of the  $L^p(\mathbb{R}^d)$  norm of Riesz transform is dimension-free for any  $2 \leq p < \infty$ , using integration by parts formula for radial Fourier multipliers. Moreover, we show that

$$\|R_j^* f\|_{L^p} \leq \left(2 + \frac{1}{\sqrt{2}}\right)^{\frac{2}{p}} \|R_j f\|_{L^p}, \text{ for } p \geq 2, \quad d \geq 2.$$

As by products of our calculations, we infer the  $L^p$  norm contractivity of the truncated Riesz transforms  $R_j^t$  in terms of  $R_j$ , and their accurate  $L^p$  norms. More precisely, we prove:

$$\|R_j^t f\|_{L^p} \leq \|R_j f\|_{L^p}$$

and

$$\|R_j^t\|_{L^p} = \|R_j\|_{L^p},$$

for all  $1 < p < +\infty$ ,  $j \in \{1, \dots, d\}$  and  $t > 0$ . Along with proved results, we will discuss possible extensions and generalizations.

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## Converse of Fuglede Theorem

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**Abstract.** The Fuglede-Putnam Theorem stands as a cornerstone in the spectral theory of bounded linear operators. One of its most notable corollaries states that the product of two commuting normal operators is also normal. The converse is not true in general, and thus, it is of interest to find certain conditions under which the normality of the product of two commuting operators implies the normality of each factor. Inspired by this problem, we investigate when subnormal operators  $T_1$  and  $T_2$  are quasinormal provided their product is quasinormal. Also, we obtain as a corollary that subnormal  $n$ -th roots of a quasinormal operator are quasinormal, and thus we answer the question asked by Curto et al. in [1]. Also, we give sufficient conditions for quasinormal (subnormal) operators  $T_1$  and  $T_2$  to be normal if their product is normal. Some results in multivariable theory of subnormal and quasinormal operators are also presented (see [2], [3]).

**Keywords:** subnormal operators; quasinormal operators; spherically quasinormal pairs; Fuglede Theorem.

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## Translational regular variation and asymptotic equivalence

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**Abstract.** Consider the class of positive functions

$$\mathcal{F} = \{x = x(t), t > a, a > 0 : x(t) \rightarrow \infty \text{ as } t \rightarrow \infty\}.$$

We say that functions  $x, y \in \mathcal{F}$  are *multiplicatively strongly asymptotic equivalent* if

$$\rho_1 = \lim_{n \rightarrow \infty} \frac{x(t)}{y(t)} = 1, \quad (0.2)$$

and *additively strongy asymptotic equivalent* if

$$\rho_2 = \lim_{t \rightarrow \infty} (x(t) - y(t)) = 0. \quad (0.3)$$

A measurable function  $f : [a, \infty) \rightarrow (0, \infty)$ ,  $a > 0$ , is *translationally regularly varying* in the sense of Karamata (see, for instance, [1]) if for each  $\lambda \in \mathbb{R}$

$$\lim_{t \rightarrow \infty} \frac{x(\lambda + t)}{x(t)} < \infty. \quad (0.4)$$

The class of such functions is denoted by  $\text{Tr}(\text{RV}_\varphi)$ .

In this paper we prove that

$$x(t)\rho_1 y(t), \quad t \rightarrow \infty, \quad (0.5)$$

implies that

$$f(x(t))\rho_2 f(y(t)), \quad t \rightarrow \infty, \quad (0.6)$$

is implied by  $f \in \text{Tr}(\text{RV}_\varphi)$ .

In connection with (0.5) and (0.6) see, for example [2].

**Keywords:** translational regular variation; additively strongy asymptotic equivalence; multiplicatively strongly asymptotic equivalence.

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## The Riesz Potential Type Operator in the Power-Weighted Variable Generalized Hölder Spaces Over a Metric Measure Space

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**Abstract.** Let  $\mathcal{X} = (\mathcal{X}, d, \mu)$  be a metric measure space, where  $d$  and  $\mu$  are the distance and measure on  $\mathcal{X}$ , respectively. To characterize  $\mathcal{X}$ , the following condition is imposed:

$$\forall x \in \mathcal{X} : \quad \mu [B(x, h)] \leq K h^N \quad \text{with} \quad h \rightarrow 0, \quad K > 0,$$

where  $B(x, h)$  symbolizes an open ball, and  $N > 0$  is real. The following integral operator is considered here as the Riesz potential type operator over an open bounded  $\Omega \subset X$ :

$$I^\alpha f(x) := \int_{\Omega} \frac{f(\sigma) d\mu(\sigma)}{d^{N-\alpha}(x, \sigma)}, \quad x \in \Omega, \quad 0 \leq \operatorname{Re} \alpha < 1.$$

This study is on the action of  $I^\alpha$  between the variable generalized Hölder spaces  $H^{\omega(\cdot)}$ , which were introduced in [1]. These spaces are defined through the following condition:

$$\omega_\Omega(f, x, t) \leq c \omega(x, t), \quad 0 < c < \infty,$$

formulated in terms of the local modulus of continuity, that is

$$\omega_\Omega(f, x, t) := \sup_{y \in \Omega \cap B[x, h]} |f(x) - f(y)|, \quad x \in \Omega, \quad h > 0,$$

or the minimum majorant of  $\omega_\Omega$  with the required properties. Specifically, we consider the conditions under which  $I^\alpha$  is bounded from

$$H^{\omega(\cdot)}(\Omega, w) := \{f : \quad w f \in H^\omega(\Omega), \quad (w f)(a) = 0\}$$

with the power weight

$$w(x) = d^\nu(x, a), \quad a, x \in \Omega, \quad 0 < \operatorname{Re} \nu \leq 1,$$

to  $H^{\omega_\alpha(\cdot)}$  characterized by  $\omega_\alpha(x, h) = h^{\operatorname{Re} \alpha} \omega(x, h)$ .

This study continues the approach of [2], where the Zygmund type estimates were obtained, and, based on these, theorems on the action of  $I^{\alpha(\cdot)}$  over a quasi-metric space were proved in the weightless and special weight cases. The results are presented in terms of the Bary–Stechkin classes, as well as the Orlicz indices.

**Keywords:** fractional calculus, Riesz potential, variable generalized Hölder space

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## Област 2

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## Гребнерове базе за модуле над Приферовим доменима

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**Апстракт.** С обзиром на то да је проблем постојања Гребнерове базе за идеал у прстену полинома над пољем, као и Нетериним доменом, решен (види [ 1]), поставља се исто питање за прстен полинома над нетериним прстеном. Домен  $R$  је Приферов домен ако је локализација  $R_P$  валуациони домен за сваки прост идеал  $P$ . Представићемо доказ да постоји Гребнерова база за коначно генерисани идеал  $I$  у прстену полинома са једном променљивом  $R[X]$ , при чему је Крулова димензија прстена  $R$  једнака 1. Овде се ослањамо на важан резултат о Приферовим доменима из [ 2]. Даље, као уопштење питања идеала, бавимо се коначно генерисаним подмодулима коначно генерисаних слободних модула над  $R[X]$ , при чему је фиксиран мономни поредак "POT" (*position over term*). Представићемо и доказ да постоји Гребнерова база и за овакве подмодуле, у случају исте претпоставке о димензији Приферовог домена  $R$ . Такође, приказаћемо и резултат да се у случају валуационог домена добија и јака Гребнерова база за наведени подмодул. Споменути резултати могу се наћи у [ 3].

**Кључне речи:** Гребнерове базе; Приферови домени; Валуациони домени.

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**Апстракт.** Декартов овал је геометријско место тачака  $M$  у равни за чија растојања од две фиксиране тачке - фокуса  $F_1$  и  $F_2$  важи релација  $m_1|MF_1| \pm n_1|MF_2| = \pm S_1$ , за параметре  $m_1, n_1, S_1 > 0$ . У раду разматрамо случај када Декартов овал постоји и није дегенерисан (случајеви елипса, хипербола или Паскалов пуж). За такве Декартове овале познато је да постоји и трећи фокус  $F_3$  колинеаран са прва два. Ова три фокуса у паровима одређују Декартов овал уз одговарајуће параметре. Полазећи од те чињенице формиран је систем полиномских једначина из кога је изведена формула за координате трећег фокуса Декартовог овала. Услови сагласности система су дати алгебарским везама међу параметрима и када важе те везе, на основу фокуса  $F_3$  и једног од фокуса  $F_1$  или  $F_2$  одређен је исти Декартов овал. Формула и услови су добијени применом псеудорезултанти, а резултати су потврђени применом Гребнерових база. Такође су разматране могуће примене у оптици.

**Кључне речи:** Алгебарска геометрија; Декартов овал; Гребнерове базе

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## Редеи-Берж симетрична функција диграфа и њене особине

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**Апстракт.** Ричард Стенли је 2022. увео симетричну функцију придружену диграфу и назвао је Редеи-Берж симетрична функција, у част двојице чувених математичара чије резултате о броју Хамилтонових путева у диграфу је успео да добије на нов начин, коришћењем техника теорије симетричних функција, [1]. У [2] је уведена комбинаторна Хопфова алгебра диграфова, и показано је да се Редеи-Берж функција може видети као слика ове алгебре при универзалном морфизму у комбинаторну Хопфову алгебру квазисиметричних функција. Дефинисан је и Редеи-Берж полином као главна специјализација ове функције. У овом излагању представимо низ особина Редеи-Берж функције и полинома које су потпуно аналогне својствима које имају хроматска функција и хроматски полином графа. Због те сличности, Редеи-Берж функција би наредних година могла да постане предмет проучавања многих математичара који се баве алгебарском комбинаториком.

**Кључне речи:** диграф; Редеи-Берж симетрична функција; хроматска функција графа.

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## On the Solvability of Certain Nonlinear Systems of Fuzzy Relation Equations

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**Abstract.** Since the pioneering works of Sanchez [ 2, 3], the systems of fuzzy relation equations have steadily taken a fundamental role in fuzzy literature, not only because they possess numerous interesting mathematical properties, but also because of their application potential. The most studied ones are the *linear systems*, in which an unknown fuzzy relation is on the one side of the equal sign [ 1]. The situation is quite different with the *weakly linear systems*. Although they can occur in many forms, roughly speaking they consist of one or more unknown fuzzy relations on both sides of the equal sign. This talk aims to cover the mathematical foundations of the described problem (cf. [ 4, 5, 6]), and deals with the solvability issues of weakly linear systems over complete residuated lattices. We provide a characterization of the set of all solutions to weakly linear systems to a desired degree. Ultimately, we demonstrate the application aspects in aggregating fuzzy relation structures.

**Keywords:** Fuzzy relation equations; Fuzzy relation inequalities; Fuzzy preorders; Fuzzy equivalences; Fuzzy relation systems.

**Acknowledgement.** The authors acknowledge the support of the Science Fund of the Republic of Serbia, Grant No. 7750185, Quantitative Automata Models: Fundamental Problems and Applications – QUAM.

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## Isogeny graphs, modular forms and signatures

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**Abstract.** Given  $p, \ell$  different primes, the *isogeny graphs*  $G(p, \ell)$  is defined as follows: the vertices are supersingular elliptic curves  $E/\mathbb{F}_{p^2}$ , the edges are  $\ell$ -isogenies. Such graphs give an easy visualization of many protocols in isogeny-based cryptography, e.g. the hash function [ 1] and the signature scheme [ 4].

It is immediate to see that from each vertex there are exactly  $\ell+1$  outgoing edges (the graph is regular). It is less obvious that, as proven by Eichler in [ 5], isogeny graphs are *connected* and have the *Ramanujan property*: the non-trivial eigenvalues fall in the Hasse interval.

We look at a generalization of these graphs, adding level structure to the elliptic curves: new interesting phenomena arise, since the graph can be  $k$ -multipartite, but up to this, we prove the Ramanujan property using modular curves.

The initial motivation for these generalizations was to prove security of a zero knowledge proof of knowledge in [ 2]: the Ramanujan property implies that random walks mix fastly, i.e. that even for not-too-long walks, the last visited vertex is close to uniformly distributed. In [ 3] we explored the problem in vast generality.

**Keywords:** Isogeny; Cryptography; Modular curves.

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## Idempotent-aided factorizations of matrices over a field

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**Abstract.** We define an idempotent-aided factorization of a matrix  $D$ , with the help of an idempotent matrix  $E$  of the same rank as  $D$ , as the representation of  $D$  in the form  $D = UV$ , where the matrix  $U$  has the same rank as  $D$  and the same null space as  $E$ , while the matrix  $V$  has the same null space as  $D$  and the same range as  $E$ . Such factorizations can be viewed as a natural generalization of full rank factorizations.

Here we provide three efficient algorithms for determining idempotent-aided factorizations of matrices over a field, as well as the fourth one that determines the so-called canonical idempotent-aided factorization. We also apply those algorithms in the construction of algorithms for testing the existence and computing group inverses and  $(B, C)$ -inverses of matrices over a field.

We also show that the concept of idempotent-aided factorization can be defined in an even more general context – in an arbitrary semigroup. We prove that every regular element of a semigroup has an idempotent-aided factorization with respect to an arbitrary idempotent from the Green's  $\mathcal{D}$ -class of that element.

**Acknowledgement.** The authors acknowledge the support of the Science Fund of the Republic of Serbia, Grant No. 7750185, Quantitative Automata Models: Fundamental Problems and Applications – QUAM

**Keywords:** Matrix factorization; Idempotent-aided factorization; Full rank factorization; Group inverse;  $(B, C)$ -inverse.

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## Simplicial complexes associated to character degrees of solvable groups

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**Abstract.** Graphs associated to the set of irreducible character degrees of a finite group  $G$  have been extensively studied as a way of understanding structure of the underlying group. Another approach, proposed by Isaacs, is to study associated simplicial complexes, namely the common divisor simplicial complex  $\mathcal{G}(G)$  and the prime divisor simplicial complex  $\mathcal{D}(G)$ . These complexes can be associated to any set of positive integers and this paper shows they are homotopy equivalent. Further, considering these complexes associated to the set of irreducible character degrees, we give a bound on the rank of the fundamental group.

**Keywords:** character theory; finite groups; algebraic topology; simplicial complex

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## Invertibility of some special matrices involving certain numbers

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**Abstract.** A family of structured matrices, such as Toeplitz, Hankel, Vandermonde, circulant, Hessenberg, Cauchy, and other well-known families of special matrices, offering extensive utility in digital signal processing, image manipulation, coding theory, statistics, quantum mechanics, and more. A particularly intriguing scenario arises when their elements are derived from  $k$ -Horadam numbers or Chebyshev polynomials, encompassing Fibonacci, Lucas, and Pell numbers. Additionally, the case where matrix entries are composed of generalized Narayana numbers is also of interest. We will present the conditions under which circulant and skew circulant matrices involving generalized Narayana numbers are invertible. These findings not only have practical applications across various fields but also contribute to advancing our comprehension of structured matrices and their inherent characteristics.

**Keywords:** circulant matrices; skew circulant matrices; invertibility; generalized Narayana numbers.

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## On the stacky Manin conjecture

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**Abstract.** A height function measures “size” of a rational point on an algebraic variety. The Manin conjecture predicts the number of rational points of bounded height on varieties satisfying the property that the inverse of their canonical line bundle is big. Remarkably, certain other counting results and predictions from different subfields of number theory, such as the Malle conjecture on the number of Galois extensions of bounded discriminant, provide very reminiscent asymptotic formulas. Recently, together with Yasuda, we have developed a version of the Manin conjecture for stacks [1] which explains the phenomenon. We will discuss the conjecture and some progress on it. The content of the talk is based on a joint work with Takehiko Yasuda.

**Keywords:** Manin conjecture; Deligne–Mumford stacks; heights.

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## An alternative to Mahler Measure of polynomials

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**Abstract.** We introduce the ratio of the number of roots of a polynomial  $P_d$ , greater than one in modulus, to its degree  $d$  as an alternative to Mahler measure. We investigate some properties of the limit ratio. We generalise this definition for a two variable polynomial  $P(x, y)$  using the Cauchy’s argument principle. We present an algorithm for calculating the limit ratio and a numerical method for its approximation. We estimated the limit ratio for some families of polynomials. Some examples of polynomials in two variables suggest a theorem for the limit ratio which is analogous to the Boyd-Lawton limit formula for Mahler measure.

**Keywords:** Mahler measure; argument principle; limit ratio.

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## Област 3

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Нумеричка математика, Дискретна  
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## On transmission irregular graphs — starlike and double starlike trees and long pendent paths

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**Abstract.** The transmission of a vertex in a connected graph is the sum of its distances to all the other vertices. A graph is transmission irregular (TI) when all of its vertices have mutually distinct transmissions. In an earlier paper, Al-Yakoob and Stevanović [ 1] gave the full characterization of TI starlike trees with three branches. Here, we improve these results by using a different approach to provide the complete characterization of all TI starlike trees and all TI double starlike trees. We subsequently implement the aforementioned conditions in order to find several infinite families of TI starlike trees and TI double starlike trees. Besides that, we disclose five families of unicyclic graphs with two pendent paths whose members are TI under certain conditions. As a direct consequence, we demonstrate the existence of TI chemical graphs of almost all even orders, thereby resolving a problem recently posed by Xu, Tian and Klavžar [ 2].

**Keywords:** graph distance; transmission irregular graph; starlike tree; double starlike tree; pendent path.

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## On the extremal Harary index of graphs with given parameters

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**Abstract.** Let  $G = (V(G), E(G))$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . The distance  $d(u, v)$  between any two distinct vertices  $u, v \in V(G)$  is the number of edges in a shortest path joining  $u$  and  $v$ . The *diameter* of a graph  $G$  is the maximum distance between any two vertices in  $G$ . A subset  $S \subset V(G)$  of mutually non-adjacent vertices in a graph  $G$  is said to be an *independent set* in  $G$ . The *independence number* is the maximum cardinality of an independent set in  $G$ .

The *Harary index* of a graph  $G$ , denoted by  $H(G)$ , was introduced in 1993, and is defined as

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}, \quad (0.7)$$

with the summation going over all pairs of vertices of  $G$ . This index was named in honor of Professor Frank Harary on the occasion of his 70th birthday. Recently, there has been a great interest in studying extremal graphs that minimize (or maximize) Harary index in different classes of graph. The graphs with diameter or independence number equal to  $n - c$ , for  $1 \leq c \leq 4$ , which attain the minimum value with respect to the Harary index are being considered and the extremal graphs are characterized, as well.

**Keywords:** Harary index; diameter; independence number.

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## Matrices in Gauss-type quadratures for variable-sign weight functions

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**Abstract.** We consider a recently proposed Gauss-type quadrature formula with respect to a weight function that changes sign in the interior of the integration interval. An important step in its construction is to introduce a modifier function used to transform the given integral into a sum of one integral that does not cause a quadrature error and the other integral with a property that the points from the interior of the integration interval at which the weight function changes sign are the zeros of its integrand. Determining a modifier function requires solving an associated system of linear equations. For the same integral, different modifier functions can be chosen, and hence different linear systems can be obtained. From a theoretical perspective, only necessary is that the associated system has a solution, but from a computational perspective, it is also important that the associated system is not too ill-conditioned and that the structure of its matrix is as simple as possible. We analyze the conditions under which it is guaranteed to obtain, for instance, a system with a Vandermonde matrix or a system with an identity matrix. We also give examples where systems with an arbitrary matrix are obtained.

**Keywords:** Gauss quadrature formula; variable-sign weight function; linear system.

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## Solving an unconstrained minimization problem using the Hybrid Modified Accelerated Gradient Method

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**Abstract.** In this paper we discuss the hybrid type of accelerated gradient method for solving unconstrained optimization problems. There are many methods in the contemporary literature developed for solving this problem. Herein, a special attention will be paid to the methods that were created as a modification of quasi-Newton's method using their hybrid versions. We specially study convergence features of the Hybrid Modified Accelerated Gradient Method, which presents a hybrid variant of the Modified Accelerated Gradient Method. This method was tested for three main properties: the number of iterations, the CPU time and the number of function evaluations. Numerical outcomes confirmed better performance profiles in favor to the derived hybrid model when compared to its forerunner.

This work was funded by the project of Faculty of Sciences and Mathematics, University of Priština in Kosovska Mitrovica (internal-junior project IJ-2303)

**Keywords:** Line search; gradient descent methods; quasi Newton method; convergence rate; hybrid model.

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## The set of anti-Gaussian quadrature rules for the optimal set of quadrature rules for trigonometric polynomials

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**Abstract.** It is well known that the anti-Gaussian quadrature rule, introduced by Laurie in 1996 ([1]), gives the error equal in magnitude but of opposite sign to the error of the corresponding Gaussian quadrature rule. Here, we define a set of anti-Gaussian quadrature rules for the optimal set of quadrature rules in Borges' sense in the linear space of trigonometric polynomials ([2], [3]). We consider the orthogonality with respect to the set of  $r$  different weight functions, with special attention to even weight functions. Also, we investigate the corresponding class of trigonometric multiple orthogonal polynomials and prove some of their important properties.

**Keywords:** anti-Gaussian quadrature rule; multiple orthogonal polynomials; trigonometric orthogonality.

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## A machine learning method with extra-gradient step

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**Abstract.** This paper deals with the minimization of unconstrained objective functions in the form of finite sums. We present an extra-gradient method with line search strategy and algorithm that uses variable sample size and thus makes the process significantly cheaper. The method is non-monotone, and the adaptive step size  $\alpha_k$  obtained in the linear search, is a random variable dependent on the sample  $\xi_k$ . The inevitable consequence is that the errors do not induce martingales. The algorithm is tested on a couple of examples, including the machine learning problems. [ 1, 2]

**Keywords:** finite sum minimization; machine learning; line search extragradient.

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## Reinforcement learning for graphs and beyond

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**Abstract.** Adam Zsolt Wagner [ 1] showed how a particular reinforcement learning technique, the so-called cross entropy method, can be used to construct (counter)examples in graph theory. We have recently provided an improved implementation of this method [ 2] and in this lecture we will showcase how it can be used to construct counterexamples for a set of older conjectures on the Laplacian spectral radius of graphs [ 2], edge-colorings of complete graphs that lead to new lower bounds on Ramsey numbers [ 3], and, with a minor adaptation, also the shape of optimal window overhangs for residential homes [ 4].

**Keywords:** Reinforcement learning; Cross-entropy method; Graph theory; Building energy optimization.

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## Iterative methods with constant and variable coefficients for computing matrix inverses

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**Abstract.** We present several iterative methods for computing matrix inverse of the matrix  $A$  as well as several generalized matrix inverses. Two classes of methods are considered:

1. methods of the form  $X_{k+1} = X_k p(AX_k)$  where  $p(x)$  is the polynomial with constant coefficients;
2. methods of the form  $X_{k+1} = X_k \left( a_0^{(k)} I + a_1^{(k)} AX_k \right)$  where  $a_0^{(k)}$  and  $a_1^{(k)}$  are coefficients computed in each iteration.

Convergence properties are studied, as well as the orders of convergence and computation efficiencies of these methods. They are verified on the several numerical examples.

**Keywords:** iterative method; generalized inverse; convergence

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## Decision-making algorithms based on fuzzy soft sets

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**Abstract.** The theory of soft sets was introduced as a mathematical tool for solving problems containing uncertainty and imprecision, as well as for the simpler representation of data with certain characteristics. Over the past twenty years, the theory of soft sets has evolved in various directions. New operations have been defined, different algorithms have been developed, all with the aim of applying the theory of soft sets in various fields. The concept of a fuzzy soft set has been studied by many, and there are numerous applications because algorithmic methods can be formulated in the theory of fuzzy soft sets, providing very useful conclusions when making decisions. Based on a given fuzzy soft set, certain numerical values can be defined as values that characterize the fuzzy soft set, and based on these values, we can formulate a decision-making algorithm.

**Keywords:** Soft set; Fuzzy soft set; Energy; Decision making.

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## Some extremal value problems of vertex-degree-based invariants on trees and connected graphs

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**Abstract.** A collection of innovative vertex-degree-based invariants, including Sombor-Index-Like Graph Invariants denoted as  $SO_i$  for  $1 \leq i \leq 6$ , and generalized reduced Zagreb indices represented as  $GRM_\lambda$  for certain  $\lambda \in \mathbf{R}$ , was recently introduced. These invariants were developed through geometric reasoning within a novel graph invariant framework. Motivated by unresolved questions and ongoing findings highlighted in [ 1] and [ 2], we determined the maximum values of  $SO_5$  and  $SO_6$ , as well as the minimum value of  $GRM_\lambda$  for  $\lambda \leq -2$ , in specific classes of trees with predefined order and maximal degree (such as molecular trees with maximal degrees of 3 and 4). Furthermore, we identified the maximum value of  $SO_5$  among the graphs resulting from applying the join operation to specific graphs of a given order.

**Keywords:** Vertex-degree-based invariants; Trees; Extremal values.

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## Metric dimension of hypercube and coin weighing problem

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**Abstract.** In this paper we consider the connection between metric dimension problem and minimal doubly resolving set problem of hypercube graph with coin weighing problem. Metric dimension of hypercube  $\beta_n$  is the minimum cardinality of a resolving set [3, 8], and  $\psi_n$  is the minimum cardinality of a doubly resolving set of hypercube graph  $Q_n$  [1]. Coin weighing problem posed by Söderberg and Shapiro can be defined as follows: for  $n$  coins, each with one of two distinct weights, determine the weight of each coin with the minimum number of weighings ( $f_n$ ) [9]. It is known that  $f_n$  differs from  $\beta_n$  by at most one [5, 7]. We have shown that it holds more precisely:  $f_n \leq \beta_n \leq f_{n-1} + 1$ . Also, we have shown that equality  $\psi_n = f_n + 1$  holds and, as a consequence,  $\beta_{m+n} \leq \beta_m + \beta_n$ . The last inequality shows that well-known hypothesis  $\beta(G \square H) \leq \beta(G) + \beta(H)$  is true in the case of hypercube graph.

**Keywords:** graph theory, metric dimension; doubly resolving set; coin weighing; hypercube graph.

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## About a class of nonlocal hyperbolic equations

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**Abstract.** Layers with material properties which significantly differ from those of the surrounding medium appear in a variety of applications. The layer may have a structural, thermal, electromagnetic or optical role, etc. The processes in domains with layers can be modelled by boundary value problems whose solutions are defined in two or more domains. In some cases these domains are disconnected. The effect of the intermediate region can be taken into account by means of nonlocal conjugation conditions. In this paper we investigate an initial boundary value problem for a one-dimensional hyperbolic equation in two disconnected intervals. In each interval an initial-boundary problem of hyperbolic type with Robin boundary condition is given, while the interaction between their solutions is described by means of nonlocal conjugation conditions. For the model problem the existence and uniqueness of its weak solution in appropriate Sobolev-like space is proved. A finite difference scheme approximating this problem is proposed and analyzed. An estimate of the convergence rate has been obtained. The problem of eigenvalues has also been considered. Theoretical results have been covered by numerical experiments.

**Keywords:** weak solution, Sobolev spaces, conjugation condition, finite-difference scheme, eigenvalue.

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## Numerical approximation of one dimensional fractional transmission problem

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**Abstract.** Fractional partial differential equations (FPDEs) have attracted significant attention in recent years, owing to their diverse applications across numerous scientific and engineering domains. Often, fractional-order models prove to be more suitable than their integer-order counterparts, as fractional derivatives and integrals facilitate the description of memory properties inherent in various materials and processes. In this context, investigation has been undertaken on a fractional-in-time transmission problem spanning two disjoint intervals. An a priori estimate has been established for its weak solution within a suitable Sobolev-like function space. The study delves into the well-posedness of an interface problem associated with this equation, demonstrating its stability within corresponding Sobolev-like function spaces. Furthermore, a finite difference scheme has been developed to approximate this problem, accompanied by a thorough analysis of its properties. An estimation of the convergence rate has been derived, aligning with the smoothness characteristics of the input data. A proposed difference scheme has been put forth and validated through several numerical examples.

**Keywords:** fractional derivative; transmission; sub-diffusion; finite differences; convergence rate.

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## Lyapunov-type extremal problem with phase constraints

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**Abstract.** We consider a nonsmooth case of Lyapunov-type, i.e. isoperimetric convex continuous-time optimization problem with inequality integral constraints and phase constraints, defined in  $L_\infty([0, T]; \mathbb{R}^n)$ . Subdifferential approximation of the considered problem proved to be a practical way to bypass the lack of differentiability. By using new alternative theorem for convex inequalities in functional spaces, necessary optimality conditions are obtained.

**Keywords:** Optimal control; Continuous-time optimization problems; Convexity; Optimality conditions;

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## Векторско паковање кутија у две и три димензије решавано Редукованом методом променљивих околина

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**Апстракт.** Паковање кутија (енгл. bin packing) познати је НП тежак проблем који се састоји у паковању задатог скупа пакета у расположиви скуп кутија. У литератури је највише заступљен проблем који укључује једну димензију кутија и пакета, а код више димензија постоје геометријско и векторско паковање. Овде је разматран векторски проблем паковања за 2 и 3 димензије. Примењена је Редукована метода променљивих околина (Reduced Variable Neighborhood Search - RVNS) која је тестирана на скуповима инстанци из литературе. Извршена су поређења добијених решења, у смислу квалитета и времена извршавања са резултатима до сада предложених метода. За поређење је коришћен и егзактни решавач уз додатно побољшање у виду боље процене горње границе потребних кутија за задату инстанцу. Резултати поређења показују да је предложена метода упоредива са постојећим приступима, а на већем броју инстанци у просеку даје боље резултате. То је охрабрујући закључак из кога следи неколико могућих праваца за унапређење ове методе. Разматрају се и могуће примене векторског паковања на реалне проблеме из праксе.

**Кључне речи:** векторско паковање кутија; метода променљивих околина; оптимизација.

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**$e$ –ПОЗИТИВНОСТ НЕКИХ НОВИХ КЛАСА ГРАФОВА**

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**Апстракт.** Правилно бојење графа је свако додељивање боја чворовима графа, такво да два суседна чвора обавезно имају различиту боју. Хроматски полином, први пут уведен 1912. године у Бирхофовом раду, броји начине на које је могуће правилно обојити граф помоћу унапред задате палете боја. Бесконачни аналогон хроматског полинома је такозвана хроматска функција, уведена од стране Стенлија 1995. године, која у себи садржи информације о свим правилним бојењима неког графа. То је једна симетрична функција, те је природно запитати се шта можемо закључити о коефицијентима у њеном развоју у различитим природним базама векторског простора симетричних функција. Сам Стенли је поставио хипотезу о позитивности ових коефицијената у елементарној бази (краће,  $e$ –позитивност) за одређену класу графова, и ово је једно од централних питања у алгебарској комбинаторици у последњих 30 година.

У овом излагању, уводимо две нове класе графова - класу сунаца и класу бучица. Показујемо како се проблем испитивања  $e$ –позитивности многих графова може свести на проблем испитивања  $e$ –позитивности сунаца, те дајемо неколико критеријума за проверу ове позитивности. Напоследку, проналазимо начин да се коефицијенти хроматске функције бучица у елементарној бази израчунају експлицитно.

**Кључне речи:** хроматска симетрична функција графа;  $e$ –позитивност; сунца; бучице.

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## Анализа и поређење варијанти проблема $p$ -хаб максималног покривања са различитим алокацијским шемама

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**Апстракт.** У овом раду су разматране различите варијанте проблема  $p$ -хаб максималног покривања неограничених капацитета ( $p - \text{HMCP}$ ). Циљ проблема је одређивање оптималних локација за успостављање унапред задатог броја хабова, тако да укупни проток међу свим покривеним паровима снабдевач-корисник буде максималан, уз претпоставку бинарног или парцијалног покривања. У случају бинарног покривања, пар снабдевач-корисник се сматра покривеним, уколико цена транспорта (време или растојање) од снабдевача до корисника није већа од унапред задате максималне цене транспорта (времена или растојања). Парцијално покривање укључује степен покривености између парова снабдевач-корисник који се реализује преко функције чија вредност опада са порастом растојања. У раду су разматране варијанте проблема  $p - \text{HMCP}$  са различитим алокацијским шемама: једноструком, вишеструком и  $r$ -алокацијском шемом ( $1 \leq r \leq p$ ). Изложене су опште математичке формулације проблема  $p - \text{HMCP}$ , раније предложене у радовима [1] и [2], а чија се предност огледа у чињеници да се лако трансформишу у формулације проблема са различитим алокацијским шемама. У раду су упоређене варијанте проблема  $p - \text{HMCP}$  кроз анализу решења добијених за различите алокацијске шеме и оба концепта покривања. За тестирање модела и анализу решења разматраних варијанти проблема  $p - \text{HMCP}$ , коришћене су стандардне  $SAB$  и  $AP$  инстанце из литературе малих и средњих димензија.

**Кључне речи:** проблем  $p$ -хаб максималног покривања; бинарно и парцијално покривање; једнострука, вишеструка и  $r$ -алокација.

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## Награда Свјетске асоцијације лауреата за Математику или Рачунарске науке

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**Апстракт.** Математичари Аркадиј Немировски, (Институт за Технологију, Атланта, САД) и Јуриј Нестеров (професор емеритус, Католички универзитет, Нови Лувен, Белгија) су добитници ове награде за 2023.годину. Награде су им уручене 6. новембра 2023.године у Шангају. Награђени су за серију пионирских радова из области конвексне оптимизације, укључујући теорију само-сагласних функција и методе унутрашњих тачака, убрзани градијентни метод. Њихови резултати се широко користе у машинском учењу. Покренули су револуцију у развоју алгоритама оптимизације. Теорија сложености метода оптимизације и убрзани алгоритми, које су они развили продубљују разумијевање метода оптимизације и могућности конструкције „оптималних метода оптимизације”.

У овом саопштењу ми ћемо кратко презентирати резултате Немировског и Нестерова у конвексној оптимизацији. Детаљније ћемо се бавити брзим градијентним методом и семидефинитном релаксацијом комбинаторних проблема.

**Кључне речи:** награда свјетске асоцијације лауреата; оптимизација.

## Раслојене фамилије функција и примене

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**Апстракт.** У недавно публикованом раду [1] разматране су реалне фамилије функција  $\varphi_p(x)$  за аргумент  $x \in (a, b)$  и реалан параметар  $p$  у случају када су монотоне по параметру. Такве фамилије имају занимљиве геометријске примене у Теорији аналитичких неједнакости [2], [3] и у раду [1] су назване раслојене фамилије функција. У таквим фамилијама је могуће, под одговарајућим условима, издвојити функцију - члан фамилије који има неке минимак особине које су од значаја за Теорију апроксимација. Такође дају се докази неких познатих и неких нових резултата у Теорији аналитичких неједнакости, а у случајевима када је то могуће одређени су и минимак апроксиманти за одговарајуће фамилије које су повезане са разматраним неједнакостима.

**Кључне речи:** Раслојене фамилије функција, минимак апроксимант.

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## Хипотеза Никифорова и случајни графови

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Сингуларне вредности графа; случајни граф; хипотеза Никифорова;

**Апстракт.** У овом раду говори се о неоријентисаним графовима без петљи и њиховом спектру. Шатенове  $p$ -норме представљају збир  $p$ -тих степена сингуларних вредности графа степенован са  $\frac{1}{p}$ , специјалан случај Шатенове норме за  $p = 1$  је баш енергија графа. Владимир Никифоров је поставио хипотезу да од свих неоријентисаних графова без петљи са  $n$  чворова највећу вредност Шатенове норме за  $p > 2$  има комплетан граф. Пошто је спектар комплетног графа познат доказивање ове хипотезе постаје доказивање одређене неједнакости. Ова хипотеза је доказана да важи за три специјалне класе графова, то су: стабла, јако регуларни графови са параметрима  $\left(n, \frac{n + \sqrt{n}}{2}, \frac{n + 2\sqrt{n}}{4}, \frac{n + 2\sqrt{n}}{4}\right)$  код којих се постиже максимална енергија и графови код којих је  $p$  паран број. Такође наводе се и нека уопштења везана за случајне графове. Нађена је горња оцена за геометријску средину сингуларних вредности случајног графа и израчуната је средња вредност  $p$ -тих степена сингуларних вредности случајног графа.

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## Област 4

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Геометрија, Топологија, Симплектичка  
геометрија, Механика



## Конструкција ауто-дуалних комплекса, ауто-дуалне триангулације многострукости

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**Апстракт.** Александеров дуал симплицијалног комплекса  $K \subseteq 2^{[n]}$  је симплицијални комплекс  $\widehat{K}^{[n]} = \{[n] \setminus A \mid A \notin K\}$ . Комплекс  $K$  је ауто-дуалан ако је  $K = \widehat{K}^{[n]}$  а под-дуалан ако је  $K \subseteq \widehat{K}^{[n]}$ . Ауто-дуални симплицијални комплекси са  $n$  темена представљају канонске примере симплицијалних комплекса који немају геометријску реализацију у простору  $\mathbb{R}^{n-3}$ , јављају се као минималне триангулације многих тополошких простора и врло су значајни у теорији комбинаторне оптимизације. Vrehm и Kühnel су 1987. у раду [1] доказали да ако  $d$ -димензионална многострукост која није сфера има триангулацију са  $n$  темена тада је:

$$n \geq 3\lceil d/2 \rceil + 3 \quad (0.8)$$

а једнакост важи само у димензијама 2, 4, 8, 16 када многострукост има хомолошки тип реалне, комплексне, кватернионске и октанионске пројективне равни. Одговарајуће триангулације су откривене, последња 2022. године у раду [2], и све су примери ауто-дуалних симплицијалних комплекса.

У [3] се истражује комбинаторна структура ауто-дуалних симплицијалних комплекса. Доказује се да за сваки ауто-дуални комплекс  $K \subseteq 2^{[n]}$  и свако теме  $\{v\} \in [n]$  важи

$$K = \widehat{\text{Lk}(\{v\})}^{[n] \setminus \{v\}} \cup C(\text{Lk}(\{v\})) \quad (0.9)$$

где је  $\text{Lk}(\{v\}) \subseteq 2^{[n] \setminus \{v\}}$  под-дуалан симплицијални комплекс у амбијенту  $2^{[n] \setminus \{v\}}$ . Једнакост (0.9) омогућава да од произвољног под-дуалног симплицијалног комплекса  $K \subseteq 2^{[n]}$  добијемо комплекс

$$\Lambda K = \widehat{K}^{[n]} \cup CK \quad (0.10)$$

(где је  $CK = K * \{\emptyset, \{n+1\}\}$ ) који је ауто-дуалан у амбијенту  $2^{[n+1]}$ . Отуда, комбинаторна и тополошка својства ауто-дуалних комплекса су у потпуности одређена комбинаторним својствима линка њиховог произвољног темена а све ауто-дуалне комплексе можемо да добијемо својеврсном надоградњом (0.10) под-дуалних комплекса.

У докторској дисертацији [4] и раду [5] се анализирају ауто-дуалне триангулације многострукости. Доказује се да се ауто-дуална комбинаторна многострукост  $K \subseteq 2^{[n]}$  димензије  $d$  добија надоградњом (0.10) под-дуалне комбинаторне сфере  $S^{d-1} \subset 2^{[n-1]}$  која је  $(n-d-2)$ -повезана тј.  $\binom{[n-1]}{n-d-2} \subset S^{d-1}$ . Користећи комбинаторну Александерову дуалност, доказује се да ауто-дуална комбинаторна многострукост  $M \subset 2^{[n]}$  димензије  $d$  не може да буде сфера и да је  $n = 3\lceil d/2 \rceil + 3$  што по резултату (0.8) Vrehm-а и Kühnel-а имплицира да је свака ауто-дуална комбинаторна многострукост димензије  $d \in \{2, 4, 8, 16\}$  и има хомолошки тип реалне, комплексне, кватернионске и октанионске пројективне равни а друге ауто-дуалне комбинаторне многострукости не постоје.

Одређују се  $f$ -вектори описаних многострукости као и  $f$ -вектори сфера чијом дуалном надоградњом се оне добијају и наводи метода за њихову конструкцију.

**Кључне речи:** Симплицијални комплекс; Александерова дуалност; комбинаторна многострукост; минималне триангулације.

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## О Јакоби-ортогоналности у Римановој и псеудо-Римановој геометрији

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**Апстракт.** У раду [ 1 ] уведена је нова потенцијална карактеризација Риманових Осерманових алгебарских тензора кривине. За алгебарски тензор кривине кажемо да је Јакоби-ортогоналан ако  $\mathcal{J}_X Y \perp \mathcal{J}_Y X$  важи за све  $X \perp Y$ , при чему  $\mathcal{J}$  означава Јакобијев оператор. У раду [ 1 ] је доказано да је сваки Риманов Јакоби-ортогоналан тензор Осерманов, док су сви познати Осерманови тензори Јакоби-ортогонални. У раду [ 2 ] је уопштен појам Јакоби-ортогоналности на недефинитне просторе са скаларним производом. Упореджени су различити принципи и утврђиване везе између Осерманових, Јакоби-дуалних и Јакоби-ортогоналних алгебарских тензора кривине. Показано је да је сваки квази-Клифордов тензор Јакоби-ортогоналан. Доказано је да је Јакоби-дијагонализабилан Јакоби-ортогоналан тензор Јакоби-дуалан кадгод  $\mathcal{J}_X$  нема изотропних сопствених вектора за свако дефинитно  $X$ . Показано је да је сваки алгебарски тензор кривине димензије 3 Јакоби-ортогоналан ако и само ако је константне секционе кривине. Доказано је да је сваки 4-димензиони Јакоби-дијагонализабилан алгебарски тензор кривине Јакоби-ортогоналан ако и само ако је Осерманов.

**Кључне речи:** тензор кривине; Јакобијеви оператори; Јакоби-ортогоналност; недефинитан простор.

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## Интеграбилни случајеви нехолономног система котрљања сфера као уопштења кретања Чаплигинове лопте

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**Апстракт.** Разматра се нехолономни систем који се састоји од  $n$  хомогених куглица истог полупречника које се котрљају без клизања између две концентричне сфере од којих је једна непокретна, а друга покретна. Покретна сфера је нехомогена, при чему се центар маса поклапа са њеним геометријским центром. Проблем је описан у четири различите конфигурације. Изведене су једначне кретања и показано је да систем поседује инваријантну меру. Једначине кретања обухватају као подслучајеве до сада познате случајеве котрљања Чаплигинове лопте по равни и сфери и њихова уопштења. У случају  $n = 1$  пронађена су два интеграбилна случаја. За одговарајући планарни проблем котрљања  $n$  хомогених лопти између две равни од којих је једна непокретна, дата је процедура интеграције у квадратурама.

**Кључне речи:** холономни системи; котрљање без клизања; инваријантна мера; интеграбилност

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## Класификација подгрупа без вишеструкости и полиномијално интеграбилни суб-Риманови геодезиски токови

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**Апстракт.** У раду [ 4] доказана је хипотеза Богојављенског о интеграбилности Ојлерових једначина дефинисаних ланцима подгрупа у случају компактних Лијевих група [ 2]. Интеграбилност је доказана у некомутативном смислу у класи полиномијалних интеграла. По хипотези Мишћенка и Фоменка треба очекивати и комутативну интеграбилност у класи полиномијалних интеграла. У случају када имамо подгрупе које су без вишеструкости (multiplicity free) или скоро без вишеструкости (almost multiplicity free) комутативни полиномијални интеграл се могу лако одредити [ 4]. Позната је класификација подгрупа без вишеструкости компактних Лијевих група [ 3], [ 6]. У [ 5] је добијена класификација подгрупа скоро без вишеструкости компактних простих Лијевих група. Резултати су примењени на суб-Риманове геодезијске токове дефинисане лево-инваријаном дистрибујом која је у неутралној групи ортогонални комплемент Лијеве подалгбре у односу на би-инваријантну Риманову метрику [ 1].

**Кључне речи:** некомутативна интеграбилност; хипотеза Мишћенка и Фоменка; подгрупе без вишеструкости.

**Табела 1.** Класификација подгрупа без вишеструкости и скоро без вишеструкости компактних простих Лијевих група

без вишеструкости	$(B_n, D_n)$	$(D_n, B_{n-1})$	$(A_n, A_{n-1} \oplus \mathfrak{u}(1))$
скоро без вишеструкости	$(A_n, A_{n-1})$	$(A_3, A_1 \oplus A_1 \oplus \mathfrak{u}(1))$	$(B_2, \mathfrak{u}(2))$
	$(B_2, B_1 \oplus \mathfrak{u}(1))$	$(B_3, \mathfrak{g}_2)$	$(\mathfrak{g}_2, A_2)$

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## Complex analytic methods for virtual properties of mapping class groups

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**Abstract.** The mapping class group  $\text{Mod}(\Sigma)$  of a closed topological surface  $\Sigma$  is the group of connected components of orientation-preserving diffeomorphisms of  $\Sigma$ . These groups are well-studied, however very little is known about their finite index subgroups. One prominent open problem in this field is a question of Ivanov: does  $\text{Mod}(\Sigma)$  admit a finite index subgroup with a finite abelianization? Since  $\text{Mod}(\Sigma)$  is the étale fundamental group of the moduli space  $\mathcal{M}(\Sigma)$  of Riemann surfaces homeomorphic to  $\Sigma$ , this question of Ivanov naturally relates to homology of finite covers of  $\mathcal{M}(\Sigma)$ . Using this approach, many results have been obtained in recent years using both Teichmüller theoretic and complex geometric methods [5, 6, 1, 7], as well as the methods of algebraic geometry [3, 2, 4]. In this talk, we will give a general introduction to the field and present some of these results.

**Keywords:** Mapping class group; Moduli space; Teichmüller space

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## Loop homology of polyhedral products and Golod rings

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**Abstract.** In the 1950s, J.-P. Serre proved that the Poincaré series of a commutative local Noetherian ring is component-wise bounded by a certain rational function depending on the Betti numbers of the Koszul complex and the minimum number of generators in the maximal ideal. In 1962, E.S. Golod showed that Serre's inequality turns into equality if and only if multiplication and all Massey products in Koszul homology of a local ring are trivial; such a local ring is called a Golod ring. J. Bakelin proved in 1982 that the Poincaré series of monomial rings are rational; among the monomial rings there is a well-known class of Stanley-Reisner rings (or face rings) of simplicial complexes.

In this talk, we will discuss how toric topology allows us to establish combinatorial, algebraic and topological conditions equivalent to Golodness and minimal non-Golodness of a face ring of a simplicial complex over any field. We are going to describe these two classes of Stanley-Reisner rings in terms of their Poincaré series, Koszul homology, and the structure of the Lie algebra on the loop homology of the corresponding moment-angle complexes. We will see how the theory of spaces with a compact torus action allows us to obtain topological interpretations of the algebraic properties of Poincaré series and Koszul homology of Stanley-Reisner rings, as well as to get some new results.

The talk is based on joint work with T.E. Panov (Moscow State University).

**Keywords:** Polyhedral product, Stanley-Reisner ring, Golod ring, Massey product, Poincaré series.

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## On the Trail of Vujičić's Coordinates-Independent Position Vector Formulation: Problematizing and Correcting Traditional Divergence and Laplacian Operators

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**Abstract.** In [ 1], where the main topic was (re-)affirmation of Vujičić's coordinates-independent formulation of a material point position vector formulation and derivation of the expressions for the tangential and normal accelerations with only second order of the polar coordinates, the problem of the inappropriateness of the traditional Divergence (and thus the Laplacian) operator(s) has been hinted, as the repercussion of the position vector representation through only the radial basis vector component. It has been indicated that the traditional formulation of the divergence operator might be inadequate for the situation where the radial symmetry is absent and the particular example of the related issue cited [ 2]. In this contribution this topic will be further elaborated and the comparison between the traditional and the rotationally-invariant/classically-covariant spatial derivatives in polar coordinates evaluated for the examples dealt with in [ 3]. While primarily the Divergence and its extension in the form of Laplacian operator will be treated, along the Gradient one, the appropriateness of traditional 'transformation' of a scalar function will be re-examined, whereby additional support will be sought for in [ 4].

**Keywords:** Differential Geometry; Spatial Derivatives; Transformation of Coordinates.

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## Quantum "dots" and non-Euclidean crystallography on the 200th anniversary of János Bolyai's absolute geometry

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**Abstract.** My about 40 years old paper [ 1] in References had got a surprising actuality in the Chemistry Nobel Prize 2023 awards for the three Laureates: Alexey Ekimov, Luis E. Brus and Mounji G. Bawendi.

Of course, the present author of that paper could not guess that time the actuality and importance that was an incidental consequence of my erroneous paper [ 2], intended to construct an infinite series of non-orientable compact hyperbolic manifolds, as a polyhedral tiling series in the Bolyai-Lobachevsky hyperbolic space  $\mathbf{H}^3$ . Fortunately, I observed and improved the mistakes soon. Namely, those constructions were not manifolds because the two fixed point orbits as punctures, where points reflections (central inversions) occur in the symmetry group of the tricky polyhedral tilings.

But these singular points, as "quantum dots" e.g. for copper and chlorine ions, respectively, in glass (silicon) fluid cause light effects (by "electron jumping-leaping") whose colours might depend on the sizes of crystal particles. That means, the mistake was much more interesting than the original intention that can be reached easily later!

**Keywords:** non-orientable compact hyperbolic manifolds; polyhedral tiling series; "quantum dots".

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## Capillary motion through a pipe with a variable cross section

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**Abstract.** Washburn's equation is one of the widely used models for describing rise of a liquid column in vertical narrow pipes. In this work, we extend the existing model by introducing the variable radius. Similar problem was recently studied in [ 2], but inertial and gravitational effects were neglected.

Governing equation is derived from the momentum balance equation in integral form, under the assumption of Poiseuille flow and no-slip boundary condition at the pipe wall. We show that asymptotic approach to equilibrium may be monotonic or oscillatory with respect to the critical parameter. This also holds true in the case of the constant radius. [ 1] We impose conditions under which certain effects (i.e. gravity, inertia or viscosity) may be neglected in the scaled equation.

**Keywords:** capillary motion, viscous fluids, asymptotic analysis

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## On generalizations of Eliashberg-Gromov's theorem

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**Abstract.** Symplectic structure on a smooth manifold is given by a differential 2-form that is closed and non-degenerate. A famous theorem of Y. Eliashberg and M. Gromov [ 1] states that the group of diffeomorphisms which preserve a given symplectic structure (i.e. the symplectomorphisms group) forms a closed subset inside the group of all diffeomorphisms equipped with the compact-open topology. This theorem had a major impact on developing of symplectic geometry, that even V. I. Arnol'd in [ 2] referred to it as the “existence theorem of symplectic geometry”.

In this talk we will try to emphasize that non-degeneracy is not necessary for obtaining  $C^0$  rigidity in the above sense. In that direction, we will prove Eliashberg-Gromov's  $C^0$  rigidity for Poisson manifolds [ 4]. More precisely, we will prove that the group of Poisson diffeomorphisms forms a closed subset inside the group of all diffeomorphisms equipped with the compact-open topology. The proof relies on the Poisson version of the energy-capacity inequality [ 3].

If time permits we will discuss other possible generalizations of Eliashberg-Gromov's rigidity.

**Keywords:** symplectic diffeomorphism;  $C^0$  rigidity; Poisson diffeomorphism; energy-capacity inequality.

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## Frobenius structure on a restricted (2+1)-TQFT

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**Abstract.** We consider certain subcategories of  $3\text{Cob}$  whose objects are collections of tori. Arrows are generated by (co)multiplication and (co)unit, together with the action of the mapping class group of torus. With the help of detected Frobenius structure our goal is to construct TQFTs for such categories. On the way we manage to distinguish some torus bundles that are not distinguished by Reshetikhin–Turaev and Turaev–Viro invariants (see [ 1] and [ 2]).

**Keywords:** (2+1)-TQFT;  $3\text{Cob}$ ; Frobenius algebra;  $3\text{Cob}$ .

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## (Non-)existence of Lagrangians in hyperkähler manifolds

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**Abstract.** The purpose of this review article is to convey two phenomena, the existence and non-existence of exact Lagrangian submanifolds inside hyperkähler manifolds that have holomorphic *contracting*  $\mathbb{C}^*$ -actions.

Lagrangians of the first kind, discovered in [1], arise as the minima of the moment map of the  $S^1$ -parts of the  $\mathbb{C}^*$ -actions. They are exact Lagrangians, which, in the example of 4-dimensional  $A_n$ -resolutions generate the (compact) Fukaya Category. The same is expected in a much bigger generality called *hypertoric varieties*.

The non-existence of other exact Lagrangians, pioneered in [2] for *ADE resolutions* (that contain the aforementioned  $A_n$ -resolutions), comes from two results: (1) vanishing of a certain Floer-theoretic invariant called *symplectic cohomology* and (2) isomorphism between this invariant and its twisted counterpart, for the different symplectic form, obtained under the hyperkähler rotation of the former one. Result (1) is generalised in [3], covering in particular all known hyperkähler manifolds with  $\mathbb{C}^*$ -actions. Result (2) is still conjectural in this generality, but if correct, it would imply for instance that there are no Lagrangian spheres for all such spaces which are not in the (lowest) dimension 4.

**Keywords:** Lagrangian submanifolds; hyperkähler manifolds; symplectic cohomology.

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## Nonlocal de Sitter $\sqrt{dS}$ gravity

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**Abstract.** Nonlocal de Sitter  $\sqrt{dS}$  gravity model is defined by the action

$$S = \frac{1}{16\pi G} \int \sqrt{R - 2\Lambda}(1 + F(\square))\sqrt{R - 2\Lambda}\sqrt{-g}d^4x$$

where  $F(\square)$  is an analytic function of the d'Alembert-Beltrami operator  $\square$  and its inverse  $\square^{-1}$ . By this way, nonlocal operator  $F(\square)$  is dimensionless. The corresponding equations of motion for the metric  $g_{\mu\nu}$  are presented.

We presented and discussed several exact cosmological solutions for homogeneous and isotropic universe. One of these solutions have properties similar to ones that are usually assigned to dark matter and dark energy. Some solutions are examples of the nonsingular bounce ones in flat, closed and open universe. There are also singular and cyclic solutions. All these cosmological solutions are a result of nonlocality and do not exist in the local de Sitter case.

Moreover, we consider Schwarzschild-de Sitter metric of the  $\sqrt{dS}$  gravity model. We present an approximative solution of linearized equation, which is related to space metric far from the massive body, where gravitational field is weak. The obtained solution is of particular interest for examining the possible role of non-local de Sitter gravity  $\sqrt{dS}$  in describing the effects in galactic dynamics that are usually attributed to dark matter. The solution has been tested on the Milky Way and the spiral galaxy M33 and is in good agreement with observational measurements.

This talk is based on joint work with Branko Dragovich, Zoran Rakić and Jelena Stanković.

**Keywords:** modified gravity; cosmological solutions; dark matter; dark energy.

## Polytopality of Bier Spheres

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**Abstract.** Extending the author's previous work on the polytopality of Bier spheres associated to threshold complexes, we establish that a simplicial complex is a threshold complex if and only if its associated canonical fan is polytopal. Furthermore, utilising experimental techniques, we show that all Bier spheres on up to 11 vertices are polytopal.

**Keywords:** Polytopes, Bier spheres, Threshold complexes.

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## Област 5

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Вероватноћа и статистика, Теорија  
информација, Анализа података





## A goodness of fit test based on the median's characteristic function

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**Abstract.** The probability density function and characteristic function of the median are explored when drawing a sample from a population with uniform or exponential distribution. The convergence of the empirical characteristic function to the population characteristic function facilitates the construction of a goodness-of-fit test, quantifying the disparity between the empirical and population characteristic functions. Through the implementation of Monte Carlo simulations, a meticulous compilation of critical values is achieved, subsequently leading to a comprehensive power analysis concerning the goodness-of-fit tests. This analysis is particularly illuminating when contextualized within the realms of both uniform and exponential distributions. The outcomes gleaned from this thorough power analysis offer compelling insights, suggesting that goodness-of-fit tests grounded in the median's characteristic function exhibit superior performance when juxtaposed against certain alternative methodologies.[ 1]

**Keywords:** characteristic function, median, goodness of fit test, MATLAB, critical values

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## Contribution of Bivariate INAR Models in Modeling Time Series of Counts

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**Abstract.** Time series of counts cover broad area of data studied by researchers from various fields of sciences. After initial development of the univariate integer-valued autoregressive models recent decade produced significant number of results in modeling bivariate time series of counts. As well as the univariate models, the bivariate models are composed of the survival and the innovation component. The dependency between the two observed processes has been achieved through the survival, the innovation or some external process, and we will discuss all three approaches. The main properties of the models are presented. Special attention is given to the challenges that arise when the models complexity is increased. The practical aspects of these models will be considered through some real-life data sets.

**Keywords:** Times series; BINAR model; Thinning operator;

## To impute or not to? A multivariate goodness-of-fit testing perspective

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**Abstract.** A multivariate normality assumption is a crucial for validity of many methods of statistical inference. Therefore, there are many proposed statistical tests for testing the mentioned assumption. However, all of the currently available tests are suitable for complete samples. When the data are not complete, i.e. some of the values are missing, one needs to adapt the existing methodology to overcome this issue. Here, we consider several approaches for usage of BHEP test for testing the multivariate normality in the context of incomplete datasets with various missingness mechanisms. We explore behavior of each of them for large sample sizes, i.e. asymptotically, as well as for small sample sizes in an extensive empirical study.

**Keywords:** missing data, empirical characteristic function, weighted  $L^2$  test

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## Estimates for the diameter of planar Brownian motion

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**Abstract.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. If  $X$  is a random variable, then the expectation of  $X$  will be denoted by  $\mathbb{E}X$  with respect to the given probability  $\mathbb{P}$ . Let  $B(t)$ , where  $t \in [0, 1]$ , be a standard planar Brownian motion. For  $0 \leq \theta \leq \pi$  we introduce the parametrized range function  $r$  given by

$$r(\theta) = \sup_{t \in [0, 1]} (B(t) \cdot e_\theta) - \inf_{t \in [0, 1]} (B(t) \cdot e_\theta),$$

with  $e_\theta$  being the unit vector  $(\cos \theta, \sin \theta)$ . We find the common distribution function  $F$  of the random variables  $r(\theta)$ . Namely, we prove that

$$F(x) = 8 \sum_{n=1}^{\infty} \left( \frac{1}{x^2} + \frac{1}{(2n-1)^2 \pi^2} \right) \exp \left( -\frac{(2n-1)^2 \pi^2}{2x^2} \right),$$

for every  $x > 0$ .

Let  $d$  be the diameter of the set  $B[0, 1]$ , that is  $d = \text{diam } B[0, 1] = \sup \{ \|B(t) - B(s)\| : t, s \in [0, 1] \}$ , where  $\|\cdot\|$  denotes the two-dimensional Euclidean norm. It is known that

$$1.601 \leq \mathbb{E}d \leq 2.355.$$

We provide better lower bound for the expected diameter of the set  $B[0, 1]$ . Namely, we have the following result

$$\mathbb{E}d \geq 1.856.$$

**Keywords:** Brownian motion; Diameter; Distribution.

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## Application of resampling methods when testing agreement with Benford's distribution

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**Abstract.** Benford's law is a statistical methodology that quickly and efficiently locates suspicious positions and figures in large dataset. This law can be used to detect anomalies, errors or frauds in different data sets, especially in financial reports of companies. Benford's law claims that leading digits from 1 to 9 appear in a decreasing logarithmic law. It means that the digit 1 appears the most frequently, followed by two, three, etc. The digit 9 has the smallest frequency according to this law. In this paper we investigate the conformity of an empirical distribution of observed real data with the Benford's distribution. In addition to the tests that are presented in the literature (z-test, chi-square test, Kolmogorov-Smirnov test, MAD test) the application of resampling methods is suggested. The main goal of the paper is to present an application of the bootstrap tests as well as the permutation tests for checking conformity with Benford's law and to indicate their comparative advantages and disadvantages.

**Keywords:** Benford's law, Data manipulation, Statistical tests, Resampling methods

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## Statistical causality and separability of stochastic processes in continuous time

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**Abstract.** In this paper we consider the statistical concept of causality for continuous-time stochastic processes. This concept unifies the nonlinear Granger causality with some of the related concepts (see [ 1, 2, 3]). This concept of causality is shown to be closely related to the notion of the extremality of measures and the martingale problem (see [ 4, 5]).

The idea of separability is to make a countable set of time points serve to determine the properties of the process. Sufficient conditions for a stochastic process to be separable are not easily formulated. What we can easily do is to go from one process, which may or may not be separable, to a separable process with the same finite-dimensional distributions.

The main results of this investigation show that separability is directly related to causality concepts. More precisely, we provide necessary conditions, in term of statistical causality, for the  $\sigma$ -algebra and the space  $L^p(\Omega, \mathcal{G}_\infty, P)$  to be separable (see [ 6]). The concept of statistical causality is related to the notion of separability of stochastic processes, especially martingales.

**Keywords:** Stochastic process; causality; extremal measure; separability.

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## On a new class of tests for the Pareto distribution using Fourier methods

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**Abstract.** We propose new classes of tests for the Pareto type I distribution using the empirical characteristic function. These tests are  $U$  and  $V$  statistics based on a characterisation of the Pareto distribution involving the distribution of the sample minimum. In addition to deriving simple computational forms for the proposed test statistics, we prove consistency against a wide range of fixed alternatives. A Monte Carlo study is included in which the newly proposed tests are shown to produce high powers. These powers include results relating to fixed alternatives as well as local powers against mixture distributions. The use of the proposed tests is illustrated using an observed data set.

**Keywords:** Empirical characteristic function; Goodness-of-fit testing; Pareto distribution;  $V$  and  $U$  statistics

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## Causality, optional and predictable projections

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**Abstract.** We consider statistical concept of causality between filtrations in continuous time which is based on Granger's definition of nonlinear causality [ 2]. It will be shown that the given concept of causality [ 1, 2] preserves some of the important properties of stochastic process when the filtration is getting larger, such as martingale, optional and predictable property.

Optional and predictable projections of stochastic processes are two important concepts in the general theory of stochastic process; they are closely related to ordinary and generalized conditional expectations. The connections between the given causality concept and the optional projection and the predictable projection of the stochastic process will be presented [ 3]. Some of the results show that the (self-)causality implies indistinguishability of the optional (or predictable) projections with respect to the considered filtrations from the ones with respect to the larger filtrations [ 3].

**Keywords:** Granger's causality; stochastic processes with continuous parameter, filtrations, optional projection, predictable projection

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## Connections between Causality and Measurable Separability of $\sigma$ -algebras

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**Abstract.** The measurable separability of  $\sigma$ -algebras is defined in [ 1]. We will give a generalization of that definition for flows of information represented by filtrations. Some of the properties of measurable separability, that are directly connected to the concept of causality in continuous time [ 3, 2], will be considered. Also, we will show some connections between measurable separability of  $\sigma$ -algebras or filtrations and other known notions from the theory of stochastic processes [ 4].

We will apply the concept of causality on Bayesian experiment to prove its measurable separability. The main question in the theory of Bayesian experiment is: may the given Bayesian experiment be 'reduced' by marginalization or by conditioning and how far does such a reduction lose no 'relevant' information. This problem will be considered in view of the principle of conditioning.

**Keywords:** filtration; causality; measurable separability; Bayesian experiment.

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## On a two-sample test for equality of matrix distributions based on Laplace transforms

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**Abstract.** Recent results concerning statistical testing in the cone of symmetric positive definite matrix distributions have mainly been focused on orthogonally invariant distributions. A goodness-of-fit test for the Wishart distribution was presented in [1], while a two-sample test for equality of orthogonally invariant distributions was studied in [2].

In this talk, our attention will be on a novel two-sample test for equality of positive definite matrix distributions, which may not necessarily be orthogonally invariant. This test is constructed as the integral of the squared difference of the empirical Laplace transforms with respect to the noncentral Wishart measure.

Additionally, we will present a power study conducted using the warp speed bootstrap method. Furthermore, we will demonstrate the applicability of the test through two real data examples. Finally, we will discuss potential generalisations of this approach.

**Keywords:** noncentral Wishart measure; Laplace transform; equality of distributions; application in finance.

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## Robust bi-objective mean-CVaR portfolio selection: applications to energy sector

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**Abstract.** A new approach to optimizing or hedging a portfolio of financial positions is presented and tested with applications to energy market. Motivated by uncertainty in the estimation of problem data we consider robust bi-objective optimization problems with mean and conditional value-at-risk objective functions where the underlying probability distribution of portfolio return is only known to belong to a certain set. To tackle the problem of uncertainty we consider two different approaches: in the first one, uncertainty is represented by an elliptic set centered at the sample estimators of mean and covariance matrix; in the second one, uncertainty takes into account experts beliefs. For both approaches, we derive analytical semi-closed-form solutions for the worst case mean-CVaR portfolio; in addition, we provide a characterization of the location of the robust Pareto frontier with respect to the corresponding original Pareto frontier.

**Keywords:** robustness; multi-objective portfolio selection; Conditional Value-at-Risk; efficient frontier; energy portfolio

## Objective analysis for generalized exponential distribution based on records with applications

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**Abstract.** This work examines an objective Bayesian analysis for Generalized exponential distribution parameters using record data. Focus was specifically on the objective priors within the record framework, including the probability matching priors, Jeffrey's prior, maximal data information (MDI) prior, and reference prior. Taking into account each prior in turn, the suitability of posterior is carefully analyzed. The estimator with the best performance indicators is found through a simulated research. Also, in this research, we study the application of Bayesian point predictors of order statistics from a future sample, which are based on the values of the  $k$ th lower record from a generalized exponential distribution. Lastly, we used an actual set of data to apply all methodologies that were provided.

**Keywords:** objective priors, interval prediction, point prediction, order statistics,  $k$ th record values, generalized exponential distribution.

## One integral transform of the copula function

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**Abstract.** We study an integral transformation of the copula function. We prove that under certain conditions, the studied integral transformation is also the copula function. Some properties of the integral transformation of a copula function are studied. Also, we derive bounds for Kendall's tau and Spearman's rho.

**Keywords:** copula function, integral transformation, dependence.

## Service rate-based analysis of two-phase single-server queueing system with hypoexponential customer service time

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**Abstract.** In this paper, we analyze the cost function of a two-phase single-server queueing system with Poisson input stream and hypoexponential customer service time. In stationary mode, the explicit form of the cost function is obtained and the points at which the function reaches an absolute minimum, are found. In the rest of the paper, additional sensitivity analysis of the optimal solutions of the cost function is done. At the end, some conclusions from the conducted analysis are presented through a comparative analysis of different forms of relationships between customer service intensities in terms of service speed changes in each phase.

**Keywords:** Poisson input stream; hypoexponential service time; cost function; optimization; sensitivity analysis.

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## On Circular Density Estimation - Fejér Kernel Approach

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**Abstract.** Density estimation is one of the most important problems in nonparametric statistics, mainly due to its applications. However, in the case of circular data, it is insufficiently explored. Here we aim to fill in this gap by exploring the use of Fejér kernel in the context of density estimation. We present some theoretical and empirical properties of such estimators, both in the classical setup, dealing with a random sample of circular data, and in the presence of the measurement error. The case of Berkson error model for circular data is considered here for the first time.

**Keywords:** Fejér polynomials; circular data; Berkson error.

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## Област 6

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Рачунарске науке





## Побољшање образовног учења: Оптимизирање генетских алгоритама с неуронским мрежама у МАТЛАБ-у

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**Апстракт.** Истраживање умјетне интелигенције обухвата двије примарне методологије: симболичку и конекционистичку, од којих свака нуди различите путеве до когнитивног моделирања. Симболични приступ проучава спознају без биолошких ограничења, док конекционистичка парадигма одражава архитектуру мозга кроз умјетне неуронске мреже. Умјетне неуронске мреже који се састоје од међусобно повезаних чворова симулирају пријенос података сличан биолошким неуронима. Овај рад се бави оптимизацијом генетских алгоритама у контексту примјене умјетне интелигенције у образовању студената користећи МАТЛАБ интерфејс. Наша студија има за циљ да прецизира апроксимације генетског алгоритама, олакшавајућ интеграцију у образовне праксе. Овај подухват обећава унапређење педагошких методологија кроз фузију техника умјетне интелигенције. Значајно, референце пружају даљи увид у сродна поља, као што су неуро-фази системи, контрола пројекта и прилагођавање параметара у симулацијама. Осим тога, радови који истражују неуро-фази контролу и класични радови из рачунарства дају информацију о смјеру нашег истраживања, доприносећи напретку умјетне интелигенције у образовању.

**Кључне речи:** Умјетна интелигенција, генетски алгоритама, образовање, МАТЛАБ, оптимизација

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## Евалуација перформанси компилатора *GraalVM* на дистрибуираном рачунарском кластеру

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**Апстракт.** У основи модерних компилатора налазе се оптимизације које утичу на време компилације и на време извршавања програма. У фази развоја, евалуација утицаја оптимизације се врши на одабраном подскупу скупа референтних програма (енг. *benchmarks*) [ 1], док се комплетна евалуација врши периодично (на пример, једанпут недељно) или тек на самом крају развојног циклуса. За извршавање евалуације се користе посебне машине са подешавањима која имају за циљ да минимизују утицај спољашњих ефеката хардвера и оперативног система на квалитет резултата саме евалуације.

У овом раду описујемо изабране пројекте у оквиру којих се развијају оптимизације за компилатор *GraalVM* [ 2] и показујемо како смањујемо време евалуације уз помоћ дистрибуираног извршавања референтних програма на рачунарском кластеру који се састоји од 100 сервера при чему сваки има од 4 до 18 језгара. За потребе делимичне евалуације током развојног циклуса, пружамо и конфигурацију која се може применити на локално окружење ради елиминације утицаја спољних фактора на квалитет резултата.

**Кључне речи:** Компилатори, евалуација, *Java*, *Graal*, дистрибуирано извршавање

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## Computer solution of an arithmetical problem

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**Abstract.** Consider the following problem from elementary algebra [1]: given the number  $n$ , determine all decimal numbers  $\overline{a_n a_{n-1} \dots a_1}$ ,  $a_n > 0$ , such that  $\overline{a_n a_{n-1} \dots a_1} = \sum_{i=1}^n a_i^n$ . For example, if  $n = 3$ , then  $371 = 3^3 + 7^3 + 1^3$ . The author gives some solutions for  $n \leq 10$ , and asks if there are solutions if  $n > 10$ . Combinatorial explosion makes the solution of this problem difficult for  $n > 10$ . Another difficulty arises because the solution demands multiprecision arithmetic. V. Janković and M. Živković [2] show that there are solutions only if  $n \leq 60$ , and reduce the problem to find the numbers  $n_j = |\{i : a_i = j\}|$ ,  $0 \leq j \leq 9$ , such that  $\sum_{j=0}^9 n_j = n$  and the number of decimal digits  $j$  of  $\sum_{j=0}^9 n_j j^n$  equals to  $n_j$  for all  $0 \leq j \leq 9$ . They list all the solutions for  $n \leq 25$ , obtained by backtracking algorithm, and show that the smallest  $n$  for which there are no solutions is  $n = 12$ . Here we give details and consider some improvements of the algorithm.

**Keywords:** multiprecision arithmetic, elementary algebra, backtracking.

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## Dynamical simulations and bisimulations between weighted finite automata over the field of real numbers

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**Abstract.** Weighted automata belong to the basic computation models in computer science. They can be understood as an extension of conventional automata in which transitions and states carry numerical or other values of different types, called weights. We will investigate simulations and bisimulations for weighted finite automata that take weights in the field of real numbers. Basic properties of simulations and bisimulations for weighted automata over real numbers will be presented. The proposed concept of bisimulations is compared with other concepts of bisimulations for weighted finite automata over the field of real numbers that can be encountered in the literature (cf. [ 1, 2, 12]).

The problem of testing the existence of simulations and bisimulations for automata over real numbers, their modeling in time-varying case and their computation is discussed. Theoretical background of the investigation is given in [ 6], in the general context of weighted finite automata over a semiring. Forward-backward bisimulation (fbb) and backward-forward bisimulation (bfb) between two weighted finite automata (WFA) require a certain system of homogeneous Sylvester equations and two vector equations. In this way, the problem is transferred to linear algebra models over the real numbers. These systems of vector and matrix equations have no solution in the general case. Our intention is to apply continuous-time dynamical systems known as the Zeroing Neural Network (ZNN) approach to solve approximately such vector-matrix systems. Zhang Neural Network or Zeroing Neural Network (ZNN) dynamics has been proposed in 2001 for solving time-varying problems. The ZNN's architecture is based on setting each element of to 0. This is accomplished using the continuous-time learning regulation that arises from the establishment of error matrix or vector equations, known as error functions. Applying block matrix representation of involved error functions in common with the vectorization and the Kronecker product, it is possible to transform the model based on systems of vector and matrix equations into a single system of linear equations with mass matrix given in block form. Then the solution of generated system of linear equations is obtained using the matrix pseudoinverse.

On the other hand, forward and backward simulation between WFA is based on certain systems of matrix and vector inequalities. Typically, such systems of inequalities possess infinite number of solutions. Usage of ZNN dynamics is usable for choosing the optimal solution. In this way, approximate solutions are considered in time-varying form. Proposed models are able to determine approximate solutions or to choose best solution between an infinite number of possible solutions.

The structure of developed ZNN models is based on composite models with various matrix or vectors error functions. Based on its global convergence without conditions, the proposed ZNN dynamical system is aimed to force the convergence of involved error functions to zero. Strategy of development of ZNN dynamics based on several error functions has been exploited in several research papers, involving [ 5, 6, 7, 8, 9].

Convergence analysis of proposed dynamical models is considered. Numerical examples are performed with different initial state matrices.

**Keywords:** Weighted finite automata, zeroing neural network, simulation, bisimulation, Sylvester matrix equations.

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## Bisimulations for weighted finite automata over the field of real numbers

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**Abstract.** Simulation and bisimulation relations are powerful tools used in many areas of computer science to match moves and compare the behaviour of various computing systems, such as labelled transition systems and automata, as well as to reduce the number of states of these systems. By moving from traditional Boolean-valued systems to quantitative ones, a need arise for both simulations and bisimulations to be quantitative, to be modeled with matrices whose entries should provide a measure of the connectivity of states of the considered systems.

Our approach consists of defining quantitative simulations and bisimulations as matrices that are solutions of certain systems of matrix inequalities and equations. Such an approach was applied for the first time in [ 4], where quantitative simulations and bisimulations between fuzzy finite automata were introduced and their basic properties were examined, while in [ 5] algorithms were developed for testing the existence of simulations and bisimulations of a given type. The same algorithms compute the greatest simulations and bisimulations, in cases where they exist. Then the same approach was applied to the study of simulations and bisimulations for non-deterministic automata [ 3], weighted finite automata over an additively idempotent semiring [ 8], and max-plus automata [ 7], as well as for weighted finite automata over an arbitrary semiring [ 6], which encompass all the previous ones. It turned out that almost identical methodology can also be applied to social networks [ 9] (in positional analysis and blockmodeling) and Kripke models of fuzzy multimodal logics [ 10, 11].

Here we use the same approach in defining simulations and bisimulations for weighted finite automata over the field of real numbers. We will present the basic properties of simulations and bisimulations for this type of weighted automata and show that there are important differences in comparison with the previously mentioned types of automata. We will also compare our concept of bisimulations with other concepts of bisimulations for weighted finite automata over the field of real numbers that can be encountered in the literature (cf. [ 1, 2, 12]). The problem of testing the existence of simulations and bisimulations for this type of automata and their computation will be discussed in a separate lecture.

**Keywords:** Weighted finite automaton, containment problem, equivalence problem, simulation, bisimulation, matrix inequations and equations.

**Acknowledgement.** The authors acknowledge the support of the Science Fund of the Republic of Serbia, Grant No. 7750185, Quantitative Automata Models: Fundamental Problems and Applications – QUAM.

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## Advance Parameter Optimization meets Electron Dose Distribution in Voxel-based Transport Simulations

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**Abstract.** Radiotherapy is crucial for treating tumors, but achieving effectiveness while minimizing damage to surrounding healthy tissue presents significant challenges. In this research, we present novel methods for automatically selecting a proper set of parameters to address these two opposing criteria: achieving maximum radiation homogeneity and minimizing exposure to organs-at-risk (OARs). Our research is based on the FOTELP-VOX program (author R.Ilić), a Monte Carlo technique that determines electron dose distribution in voxel-based transport simulations utilizing patient anatomy obtained from CT images.

Researchers utilize simulations to test various scenarios in radiation therapy to mitigate potential health consequences for patients. Finding the optimal scenario for each patient is crucial yet time-consuming, often relying on a manual trial-and-error approach with loose guidelines. This type of problem is well-recognized and falls within the class of optimization problems such as the traveling salesman and scheduling.

We enhance the current methodology using standard optimization techniques like random search, as well as advanced techniques including Bayesian optimization (BO) and genetic algorithms (GA). Our goal is to efficiently search the parameter space to find the closest solution to the existing AAA electron dose calculation model.

**Keywords:** Voxel-based simulations; evolutionary optimization; Bayesian optimization; Monte Carlo techniques

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# Hyperchaotic Systems and Other Mathematical Constructs for Enhanced Image Cube Encryption

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**Abstract.** This research work introduces an encryption algorithm for image cubes, which is underpinned by hyperchaotic systems formulated from complex differential equations. The sensitivity to initial conditions, a fundamental aspect of chaos theory, has been expanded into the hyperchaotic domain to facilitate effective multidimensional encryption. The algorithm's complexity has been enhanced by the integration of Linear Feedback Shift Registers (LFSRs) and DNA coding sequences. Secure pseudorandom sequences are provided through the use of LFSRs, and additional cryptographic depth is introduced with DNA coding. This combination has resulted in a robust encryption mechanism that ensures the confidentiality of data and resilience against advanced computational threats. Superior performance in entropy, key sensitivity, and resistance to statistical attacks has been demonstrated by the proposed encryption approach, establishing its suitability for the protection of volumetric image data [ 1, 2].

**Keywords:** Chaos Theory; Cryptography; DNA Coding.

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## A calculator of some special mathematical functions

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**Abstract.** This paper presents the implementation of a calculator of certain special mathematical functions in the form of an efficient web application with a simple and intuitive GUI (graphical user interface). The objective of this application is to enable accurate numerical approximations of the most frequently used special mathematical functions in engineering and science, eliminating the necessity to acquire additional notational knowledge or programming languages syntax experience. The investigation can be divided into three larger units. The initial section offers a theoretical introduction – the functions whose approximations are implemented are defined and the formulas used for their approximation are given. For the time being, this application provides approximations for the following special mathematical functions: Bessel functions of the first kind, gamma and beta functions, and some orthogonal polynomials – Legendre, Laguerre, Hermite (physicist’s and probabilistic) polynomials, Chebyshev polynomials of the first and the second kind, as well as Jacobi polynomials. The middle section presents a description of the computer implementation – an overview of the used technology and implemented algorithms, while final section includes an overview of the application and discussion of the solution, as well as a comparison with existing software which provides same features as the subject application. At the end, it is pointed on improvements regarding existing software, as well as on development directions.

**Keywords:** special functions, orthogonal polynomials, gamma function, numerical approximations, numerical calculator

## Beyond simple thresholding: using tracklet information to improve selection of true positive detections for multiple object tracking

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**Abstract.** State of the art multiple object tracking (MOT) algorithms typically follow tracing by detection paradigm, in which detection and association are performed in separate steps. Selecting only true positive detected bounding boxes for association is one of the problems which remains largely unsolved. Existing methods usually rely on simple thresholding to discard detected bounding boxes with low detection confidence scores and recent methods (e.g. [ 1, 2]) have applied two-stage association to use low-confidence detections and remaining tracklets in the second stage. However, recent papers discussed the shortcomings of the two-stage association [ 3, 4]. In our research, we expand the idea of boosting detection confidence of likely objects from BoostTrack [ 4]. We experiment with different similarity measures in addition to intersection over union (shape, Mahalanobius distance, visual embedding) for increasing the detection confidence of bounding boxes where an object is likely to be. Furthermore, we introduce varying thresholds for increasing the detection confidence depending on the number of timesteps elapsed since the last association. We perform experiments on MOT17 [ 5] and MOT20 [ 6] datasets to demonstrate the effectiveness of our methods and achieve improvements in the MOTA metric indicating that our method increases the number of true positive detections used.

**Keywords:** multi-object tracking; detection confidence; tracking by detection.

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## Modeling the Correlations of Coronavirus and Codon Using Measures

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**Abstract.** The large number of sequenced isolates of the coronavirus family represents a massive sample for various bioinformatics experiments, including research of genomic variability and research of different data modeling techniques. The goal of our research was to use data mining techniques to determine correlation between codon usage and different types of viruses and proteins. The material includes 980,554 isolates with 15,573,303 coding sequences (proteins) of 7 coronavirus types. Material was downloaded from NCBI (26.05.2023.). Individual coding sequences containing ambiguous nucleotide codes were eliminated. As a measure of codon usage various measures (RSUC, ENC, RCBS, codon frequencies, and others) were used. We have tested different classification and clustering algorithms to construct models based on the used measures. The results we obtained showed that codon measures can be used to construct prediction models that predict the type of virus or protein with very high accuracy (from 96.3% to 99.9%). The clustering of the material used led to a separation of records that is very close to the natural clustering by protein type.

**Keywords:** bioinformatics; data mining; coronavirus; classification; clustering.



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## Област 7

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Методика наставе математике и  
рачунарства

Настава математике и информатике





## Оцењивање ученика у настави математике (педагошка пракса)

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**Апстракт.** Оцену треба схватити као резултат упознавања развоја и вредновања знања (укључујући умења и навике), залагања и активности ученика. Оцена из математике треба да изрази стварни успех појединог ученика у учењу већ треба да буде ваљана (да показује у којем степену је ученик усвојио прописани обавезни програм), објективна (да зависи само од показаних постигнућа, а не од субјективних утисака наставника или природе инструмената којим се врши оцењивање) и поуздана (да се за исти степен усвојености програмских садржаја у поновљеним оцењивањима вђод другог наставника или неком објективном методом вђодобије иста оцена). У раду се методолошки из педагошке праксе излажу облици и методе оцењивања, организација, утврђивање оцена, критеријум и норме оцењивања, о проблемима оцењивања у педагошкој пракси и личности наставника.

Биће изложени примери бод система оцењивања на основу квалитета рада ученика у погледу развоја критичког мишљења и усвојености знања стеченог на основу разумевања и примену у погледу решавања проблемских ситуација из живота.

Посебно ће бити речи са примерима о проблемима оцењивања у наставној пракси, и експерименталног бод система оцењивања у настави у ОШ К. Петровић с. Крушевица и ОШ Б. Миленковић с. Шишава, СО-е Власотинце, Србија. Ову тему са личним експериментом сам излагао на два математичка конгреса, Педагошка пракса листа Просветни Преглед и публиковао у методској књизи МАТЕМАТИКО МОЈА (Педагошка пракса).

**Кључне речи:** оцењивање; оцена; успех; критеријум; прилог.

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## Шта је Математички талент?

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**Апстракт.** Математички талент је јединствен математички портал који је веома користан за наставнике и наставу математике.

Портал је доступан на адреси: <https://matemackitalent.mk/>

Портал „Математички талент” садржи целине као што су наставни програми, чланци, књиге, такмичења..., а сваки од ових делова је систематично подељен по математичким дисциплинама (алгебра, геометрија, теорија бројева, комбинаторика, историја математике...) и дат је на бар два - три нивоа (ученици млађих разреда основне школе, ученици старијих разреда основне школе, средњошколци).

Портал је намењен обдареним ученицима и наставницима који са њима раде, што подразумева велике могућности коришћења материјала за самостални рад ученика, додатну наставу математике, припрему за математичка такмичења и друге облике рада са талентованим ученицима. Садржај портала може бити веома користан и за студенте математике, редовну наставу математике, стручно - педагошко усавршавање наставника...

Аутор, алфа и омега портала је проф. др Ристо Малчески из Скопја.

Циљ овог саопштења је да се портал „Математички талент” приближи слушаоцима овог саопштења и препоручи за употребу у разним наставним и ваннаставним ситуацијама, да се укаже на отвореност портала за нове сараднике и њихове прилоге и упути најшири позив ученицима, студентима и наставницима да користе портал.

## Методика наставе математике у специјализованим одељењима ОШ при МГ - 20 година искуства

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**Апстракт.** У овом кратком излагању ћемо представити приступе и савремене методе рада на редовној настави, као и на додатној настави, у раду са талентованим ученицима, који похађају 7. и 8. разред у специјализованим одељењима основне школе при Математичкој гимназији.

Такође, представићемо и коментарисати њихова постигнућа, а посебно успехе на завршним домаћим и међународним такмичењима.

На крају ћемо дати неке конструктивне предлоге у циљу побољшања услова рада наставника и ученика.

## Да ли наставу математике треба осавременисти?

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**Апстракт.** У настави математике у Србији, а чини се и у окружењу, већ деценијама се ништа значајно не мења и наставом (по свој прилици) и даље царују фронтални облик рада, предавачка метода и наставни програми који су само формално од образовно-васпитних циљева и задатака усмерени ка стандардима, образовним нивоима и исходима (чију реализацију углавном нико не проверава).

У међувремену се десило много тога, што је ситуацију у образовању и у настави математике, у свету, а и код нас, генерално променило. Велика промена се одиграла у сфери употребљивости математичких знања где више никог не интересује репродуктивна моћ ученика (коју и даље форсирају наша екстерна проверавања и истраживања), већ функционалне математичке способности и решавања проблема (које испитују међународна истраживања). Не мање промене су присутне у сфери наставних технологија (метода и облика рада), где пасивне и репродуктивне односе све више замењују активно учење и развијање креативности и где је присутно прилично много позитивних новина. Највеће промене су се ипак десиле у сфери наставних средстава, где је применом савремених информационих технологија могућа много очигледнија и ефикаснија настава која чини да се у јединици времена постигне много више него у класичној настави. Ако се свему претходном дода и неопходност промена у мотивацији ученика, јер је претходно најчешће коришћен мотив - оцена, у међувремену (чини се некритички и непотребно дијаметрално) изгубио значај, онда постаје јасно да је време за разматрање свих наведених и још неколико ненаведених проблема.

Своја размишљања не бих излагао као категоричне ставове, јер су ми за сваки од наведених сегмената потребна егзактна истраживања којим би их поткрепио, али мислим да би одговори на неколико питања дали значајан допринос теми о којој је реч:

1. Имамо ли снаге, средстава и људи да компаративним истраживањима стања наставе математике код нас и у свету, одредимо нашу тренутну позицију?
2. Како организовати рад на истраживањима, анализама и коришћењу добијених резултата?
3. Како на основу добијених истраживања и њихових резултата одредити у ком смеру треба да се крећу промене када се говори о стручним, али и дидактичко-методичким аспектима наставе математике?
4. Како осмислити и организовати систематичан рад на реализацији претходно набројаних послова?
5. Како обезбедити да стратегија промена као резултат учињених истраживања буде и реализована, али и континуирано и системски праћена и унапређивана (акциони план, неопходна средства, институције, кадрови ...)?

## Развој и имплементација образовне платформе еВежбаоница

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**Апстракт.** Министарство просвете и Завод за вредновање квалитета образовања и васпитања, у оквиру заједничког пројекта, а у сарадњи са еУправом, Математичким факултетом и Факултетом организационих наука Универзитета у Београду, креирали су бесплатну и јавно доступну платформу еВежбаоница. Ова платформа омогућава ученицима да на свом матерњем језику вежбају и проверавају своје знање током припрема за завршни испит на крају основног образовања и васпитања. Платформа садржи задатке из седам предмета (српски језик / матерњи језик, математика, физика, хемија, биологија, географија и историја), распоређене на три нивоа тежине (основни, средњи и напредни). Преко 4000 задатака састављено је у складу са образовним стандардима на српском језику и осам језика националних мањина. Платформа еВежбаоница је развијена коришћењем модерних веб технологија и прилагодена је различитим уређајима, укључујући рачунаре, таблете и паметне телефоне. За потребе платформе развијен је специјализован језик за креирање различитих типова задатака и дефинисање њиховог визуелног изгледа. Овај језик омогућава једноставан унос различитих типова задатака (вишеструки избор, кратак одговор, избор у табели итд.) и не ограничава ауторе да приликом осмишљавања задатака користе унапред дефинисане шаблоне. Поред тога развијен је текст процесор који у реалном времену компајлира унети садржај на дефинисаном језику и приказује изглед задатка намењен ученику. еВежбаоница је веома добро прихваћена од стране ученика али и шире јавности и представља значајан корак ка дигитализацији образовања у Србији, нудећи бесплатне и доступне образовне ресурсе за ученике и подржавајући различите стилове учења и потребе ученика у процесу припреме за завршни испит.

**Кључне речи:** еВежбаоница; завршни испит; образовна платформа.

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## Површина и запремина - неки интересантни феномени

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**Апстракт.** У овом излагању размотрићемо појмове површина фигуре у равни, површина фигуре у простору и запремина тела у простору. Видећемо како се одговарајуће формуле изводе коришћењем интеграла, али и како се неке од њих могу извести елементарно. Указаћемо на неке интересантне феномене које се могу уочити приликом извођења ових формула.

**Кључне речи:** Површина фигуре; површина површи; запремина тела.

## Диференцијалне једначине и могућност примене ChatGPT-а у настави математике

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**Апстракт.** Промене у наставним плановима и програмима доводе до појаве нових изазова у на-ставном процесу. Један од таквих изазова је и недостатак адекватне литературе за поједине изборне предмете. Према наставном плану и програму за четворогодишњи образовни профил електротехничар информационих технологија [1], у оквиру изборног предмета Изабрана поглавља математике у четвртог разреда средње стручне школе обрађује се и тема Диференцијалне једначине [2]. Садржај ове теме углавном није у целости обухваћен актуелним уџбеницима који се користе у настави. Применом појединих онлајн алата доступних како наставницима тако и ученицима, могуће је унапредити на-ставу математике и донекле превазићи постојеће изазове. У складу са тим, имајући у виду рапидан развој вештачке интелигенције, у овом раду разматран је и потенцијал ChatGPT-а у пружању додатне подршке ученицима приликом усвајања садржаја из области Диференцијалних једначина.

**Кључне речи:** диференцијалне једначине; настава математике; ChatGPT .

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## Проблеми размере у основном и средњем образовању

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**Апстракт.** Изучавање размере почиње у основној школи, а размера и пропорција се појављују у већини математичких садржаја, током целог школовања ученика. Поред математичких, размера и пропорција се појављују и у наставним садржајима других предмета, као што су географија, техничко и информатичко образовање, физика, хемија, стручни предмети у средњим школама, итд. Због тога се, код изучавања размере, често примењују проблемски задаци, за чије је решавање потребно применити, не само математичко, него и знање других предмета. Упркос њиховој не малој заступљености у настави, значајан број ученика има потешкоће у примени размере и пропорције у решавању реалних проблема. Примена савремене технологије у настави омогућава да се изучавању размере приступи кроз решавање реалних проблема, какви, без примене одговарајућих софтверских решења, не би били примерени настави математике у основном и средњим школама. Истраживање, које је спроведено у Гимназији "Светозар Марковић" у Новом Саду, показало је да примена одговарајућих проблема, који ће бити представљени учесницима Конгреса, у окружењу динамичког софтвера, доприноси бољој мотивисаности ученика за изучавање размере, али и других (не само математичких) наставних садржаја.

**Кључне речи:** Динамички софтвер; проблемски задаци; размера.

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## Курикулум математике у нашој земљи и окружењу

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**Апстракт.** На почетку излагања осврнућемо се на различите дефиниције курикулума у образовању и на различита шватања курикулума. Навешћемо кратак историјат развоја и шватања курикулума. Набројаћемо главне особине курикулума у савременом образовању и продискутовати о њима. Централни део излагања односиће се на упоређивање курикулума у земљама у окружењу. Шта можемо применити у нашем образовном систему? Колико су курикулуми прилагођени новим програмима наставе и учења заснованим на ишодима? Како на најбољи начин наставу математике прилагодити потребама савремених курикулума? У сваком курикулуму нагласак се ставља на циљеве, задатке и садржаје који се морају операционализовати, реализовати и учинити мерљивим. Размотрићемо колико у њему долазе до изражаја карактеристике модерне наставе (диференцијација, пројектна настава, коришћење ИКТ – а). Упоредићемо присуство ових карактеристика у нашој земљи и земљама у нашем окружењу. Посебан осврт ћемо направити на садржаје који се могу применити при раду са даровитом децом. Да ли су они заступљени у довољној мери? А у околним земљама? Даћемо предлоге како се садржаји из курикулума могу прилагодити потребама деце која раде по ИОП – у 3. Показаћемо примере како се поједини садржаји курикулума могу додатно приближити ученицима коришћењем ИКТ – а. Посебан део излагања односиће се на присуство вероватноће у курикулуму и почетку њеног изучавања у образовању. Мишљења смо да са изучавањем треба почети већ у 5. разреду основне школе. Навешћемо примере који су „згодни” за почетно изучавање основа вероватноће. Део курикулума представљају и садржаји предвиђени за секције и ваннаставне активности. Показаћемо примере који се могу користити за ваннаставне активности и секције.

**Кључне речи:** курикулум, диференцијација, ИОП -3, вероватноћа.

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## Hyperbolic geometry, geometry of the Minkowski space, the Möbius transformations and the special theory of relativity

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**Abstract.** We review basic property of the flat geometry based on everyday intuition which is called Euclidean geometry (or parabolic geometry), and the non-Euclidean geometries which is called hyperbolic geometry (or Lobachevsky-Bolyai-Gauss geometry), and Minkowski geometry which is related Einstein's Postulate 2: The speed of light in vacuum is the same for all observers, regardless of the motion of light source or observer as well connections between these geometries.

In particular, we try to make a mathematical model motivated by the special theory of relativity, without using the second postulate of Einstein, and to consider it. We try to conclude that the two way speed of light is invariant.

**Keywords:** The Minkowski space; the special theory of relativity

## Two interesting triangle geometrical problems

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**Abstract.** First problem statement: Let ABC be the triangle with angle at the vertex A equal to  $60^\circ$ . Let H be the orthocenter of that triangle, P and Q centers of length BH and CH, respectively, M the intersection of the normal on BH at point P with line AB and N intersection of the normal on CH at point Q with line AC. Prove that the points M, H, N, O belongs to the same line.

Second problem statement: The bisector of the interior angle at vertex A of triangle ABC intersects BC at point D. If  $AD = AB$  and AD is perpendicular to OH, where O is the centre of the circumscribed circle and H is the orthocentre of that triangle, calculate triangle angles.

Both problems were on Serbian mathematical competitions, and appeared to be not so easy but with nice pure geometrical solutions. In solutions we used simple geometrical properties: triangle congruencies, central and peripheral angle in the circle, simple angle chase, bisector of isosceles triangle.

**Keywords:** triangle, orthocenter, bisector, circumscribed circle.

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## Enhancing C++ Programming Skills through Collaborative Peer Feedback

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**Abstract.** Student collaboration is increasingly emphasized in modernized informatics curricula to achieve competency-based learning outcomes. This evolution challenges the traditional individualistic approach in education and requires a revision of assessment practices and feedback to ensure constructive alignment. Considering that project collaboration is particularly emphasized in the field of informatics, significantly affecting later work in industry, it is crucial for students to become familiar with collaborative work during their studies and learn to give and to respond suggestions adequately.

In this study, students initially worked individually on various projects applying basic object-oriented programming concepts in C++. Following this, they were divided into groups of three, where each member reviewed and provided feedback on two projects, encouraging a collaborative learning environment. This setting has proven to provide valuable formative feedback on professional behavior and programming skills in C++, while also presenting the challenge of understanding "foreign code".

Even though 64% of students were new to commenting on other's work and providing suggestions, a remarkable 88% of them later made corrections to their work based on the received feedback. Providing formative peer feedback early and often supports the growth of crucial competencies in informatics education, like self-regulated learning essential for teamwork.

**Keywords:** peer feedback; object-oriented programming; education; foreign code analysis.

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## One-to-one mapping as a key factor in understanding the cardinality of sets among engineering students.

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**Abstract.** The use of different task formats (related to the same concept) has proven to be a good strategy for eliciting cognitive conflicts and opportunities for deeper analysis of responses. Through its application, we attempted to explore how engineering students deal with tasks related to the concept of set cardinality. The study involved 269 students from two universities: the University of Belgrade and the University of Novi Sad. By analyzing students' responses, we discovered the most common misconceptions in solving such tasks and why understanding the concept of functions (one-to-one mappings) played a crucial role. The results of this research indicate the need for a thorough treatment of the concept of functions in elementary and high school, as understanding this concept serves as a crucial foundation for further acquisition of more complex mathematical concepts.

**Keywords:** cardinality of sets; one-to-one mapping; misconception; engineering students;

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## Learning Mathematics Using Chatbots

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**Abstract.** AI-based applications named Chatbots could be used in education to arrange personalized learning activities and provide time-saving teaching supports (e.g., [ 1]). Their users interact with them in the following way: the user asks a Chatbot questions and clarifications and receives from it human-like answers and suggestions. Focusing on specific aspects of mathematical, statistical, and computational thinking [ 2, 3, 4], this contribution examined how Chatbots could be used to support the learning of mathematics. It was found that when prompted with clear and specific instructions, the applied Chatbot could solve a variety of tasks (e.g., analyze given graphs in statistical terms, solve a geometry task that connects the elements of an isosceles triangle, explain how to calculate the square root of a number using approximations). The examination showed that, in general, Chatbots can solve problems in steps (demonstrate decomposition & algorithmization), derive a common pattern from concrete examples (act as if abstracting & generalizing), or explain how to outsource certain computations to computer programs (give algorithmization & automation). However, some solutions may be too general, just partial, or even wrong, requiring the submission of carefully crafted additional prompts – a challenging task for most students. It is thus important to examine the influence of this novel kind of learning on students' achievement in mathematics and their self-confidence in learning it (e.g., [ 5]).

**Keywords:** Chatbot; learning; mathematics.

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## Using GeoGebra for Enhanced Visualization of Economic Functions

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**Abstract.** For years, GeoGebra has been established as a powerful tool for learning topics related to functions. Numerous studies have proposed the implementation of calculus teaching using GeoGebra effectively, along with the effects of such teaching on students' achievements and motivation. Over the years, we have observed that students enrolled in the Introduction to Financial Mathematics course fail to recognize the connection between the properties of real functions they explored during their previous mathematical education and the concepts of economic functions, where significant number of problems are solved using the appropriate properties of linear, quadratic, and other polynomial functions. In line with this observation, a methodological approach was devised based on the use of the GeoGebra software package for visualizing economic functions, aiming to help students apply their previously acquired mathematical knowledge more successfully and efficiently in solving specific problems related to economic functions. Research results indicate that students have made noticeable progress in applying knowledge related to determining zeros, signs, monotonicity, extreme values of functions, etc., in the context of solving concrete problems related to economic functions. In this way, students have not only reinforced their existing knowledge but have also become proficient in interpreting it in the given context.

**Keywords:** GeoGebra; visualization; economic functions; properties of a function.

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